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Unified Treatment for Harmonic Univalent Functions *

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Abstract

The authors introduce and study a unified class of $\mathcal{T}_{\mathcal{H}}(\alpha, \beta, n)$ of starlike and convex functions of order α in the open unit disk. A number of results obtained which include the coefficient estimates, sharp distortion theorems, and modified Hadamard products of functions belonging to the class $\mathcal{T}_{\mathcal{H}}(\alpha, \beta, n)$.

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1. Introduction

Denote by $\mathcal{H}$ the class of functions $f = h + \bar{g}$ that are harmonic univalent and sense-preserving in the unit disk $\Delta = \{z : |z| < 1\}$ for which $h(0) = f(0) = f_z(0) - 1 = 0$. Then for $f = h + \bar{g} \in \mathcal{H}$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n. \tag{1}$$

Note that if the co-analytic part of f that is g is zero, then $\mathcal{H}$ reduces to the class of normalised analytic univalent functions. For $0 \leq \alpha < 1$ we let $\mathcal{S}_{\mathcal{H}}(\alpha)$ denote the subclass of $\mathcal{H}$ consisting of harmonic starlike functions of order α. A function f of the form (1) is said to be harmonic starlike of order α, for $|z| = r < 1$ (see Sheil-Small [[4],p.244]) if

$$\frac{\partial}{\partial\theta}(arg f(re^{i\theta})) \geq \alpha.$$

Further denote by $\mathcal{T}^*_{\mathcal{H}}(\alpha)$ the subclass of $\mathcal{S}_{\mathcal{H}}(\alpha)$ such that h and g in $f = h + \bar{g}$ are of the form

$$h(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n. \tag{2}$$

Recently, Jahangiri[1] defined the class $\mathcal{T}^*_{\mathcal{H}}(\alpha)$ of the form (2) which satisfies the condition

$$Re\left(\frac{zh'(z) - z\bar{g'}(z)}{h(z) + \bar{g}(z)}\right) > \alpha, (0 \leq \alpha < 1). \tag{3}$$

Also, Jahangiri [1] proved that if $f = h + \bar{g}$ given by (1) and if

$$\sum_{n=1}^{\infty}\left(\frac{n-\alpha}{1-\alpha}|a_n| + \frac{n+\alpha}{1-\alpha}|b_n|\right) \leq 2, 0 \leq \alpha < 1, a_1 = 1, \tag{4}$$

then f is harmonic, univalent and starlike of order α in Δ. This is proved to be also necessary if $f \in \mathcal{T}^*_{\mathcal{H}}(\alpha)$. In fact, an interesting recent work on harmonic close-to-convex functions has seen in [3], which was associated with the Alexander integral transform.

2. Coefficient Inequality

Our main tool in this paper is the following result given by Jahangiri[1] as mentioned above.

Lemma 2.1. *Let the function $f = h + \bar{g}$ be given by (2). Then $f \in \mathcal{T}^*_{\mathcal{H}}(\alpha)$ if and only if*

$$\sum_{n=1}^{\infty}\left(\frac{n-\alpha}{1-\alpha}|a_n| + \frac{n+\alpha}{1-\alpha}|b_n|\right) \leq 2, \tag{5}$$

where $0 \leq \alpha < 1$ and $a_1 = 1$.

Next, by observing that

$$f \in \mathcal{C}_{\mathcal{H}}(\alpha) \Leftrightarrow zf'(z) \in \mathcal{T}^*_{\mathcal{H}}(\alpha), \tag{6}$$

we gain the following Lemma 2.2.

Lemma 2.2.[2] *Let the function $f = h + \bar{g}$ be given by (2). Then $f \in \mathcal{C}_{\mathcal{H}}(\alpha)$ if and only if*

$$\sum_{n=1}^{\infty}\left(\frac{n(n-\alpha)}{1-\alpha}|a_n| + \frac{n(n+\alpha)}{1-\alpha}|b_n|\right) \leq 2, \tag{7}$$

where $0 \leq \alpha < 1$ and $a_1 = 1$.

In view of Lemma 2.1 and Lemma 2.2, the unification of the classes $\mathcal{T}^*_{\mathcal{H}}(\alpha)$ and $\mathcal{C}_{\mathcal{H}}(\alpha)$ arises naturally and so a new class $\mathcal{T}_{\mathcal{H}}(\alpha, \beta, n)$ is introduced. Thus we say that a function f defined by (2) belongs to $\mathcal{T}_{\mathcal{H}}(\alpha, \beta, n)$ if and only if,

$$\sum_{n=1}^{\infty}[(1-\beta+n\beta)(n-\alpha)|a_n| + (1-\beta+n\beta)(n+\alpha)|b_n|] \leq 2(1-\alpha), \tag{8}$$

Clearly, we obtain

$$\mathcal{T}_{\mathcal{H}}(\alpha, \beta, n) = (1-\beta)\mathcal{T}^*_{\mathcal{H}}(\alpha) + \beta\mathcal{C}_{\mathcal{H}}(\alpha),$$

so that

$$\mathcal{T}_{\mathcal{H}}(\alpha, 0, n) = \mathcal{T}^*_{\mathcal{H}}(\alpha),$$

and

$$\mathcal{T}_{\mathcal{H}}(\alpha, 1, n) = \mathcal{C}_{\mathcal{H}}(\alpha).$$

3. Growth and Distortion Theorem

A distortion property for function f in the class $\mathcal{T}_{\mathcal{H}}(\alpha, \beta, n)$ is given as follows:

Theorem 3.1. *Let the function f defined by (2) be in the class $\mathcal{T}_{\mathcal{H}}(\alpha, \beta, n)$, $|z| = r < 1$, then*

$$|f(z)| \leq (1 + |b_1|)r + \left(\frac{1-\alpha}{(1+\beta)(2-\alpha)} - \frac{1+\alpha}{(1+\beta)(2-\alpha)}\right)r^2 \tag{9}$$

and

$$|f(z)| \geq (1 - |b_1|)r - \left(\frac{1-\alpha}{(1+\beta)(2-\alpha)} - \frac{1+\alpha}{(1+\beta)(2-\alpha)}\right)r^2. \tag{10}$$

The bounds in (9) and (10) are attained for the functions f given by

$$f(z) = (1 + |b_1|)\bar{z} + \left(\frac{1-\alpha}{(1+\beta)(2-\alpha)} - \frac{1+\alpha}{(1+\beta)(2-\alpha)}\right)\bar{z}^2$$

and

$$f(z) = (1 - |b_1|)z - \left(\frac{1-\alpha}{(1+\beta)(2-\alpha)} - \frac{1+\alpha}{(1+\beta)(2-\alpha)}\right)z^2$$

for $|b_1| \leq \frac{1-\alpha}{1+\alpha}$.
Proof.

Let $f \in \mathcal{T}_{\mathcal{H}}(\alpha, \beta, n)$. Taking the absolute value of f we obtain

$$\begin{aligned}
|f(z)| &\leq (1+|b_1|)r + \sum_{n=2}^{\infty}(|a_n|+|b_n|)r^n \\
&\leq (1+|b_1|)r + \sum_{n=2}^{\infty}(|a_n|+|b_n|)r^2, \\
&= (1+|b_1|)r + \frac{1-\alpha}{(1+\beta)(2-\alpha)}\sum_{n=2}^{\infty}(1+\beta)(\frac{2-\alpha}{1-\alpha}|a_n| + \frac{2-\alpha}{1-\alpha}|b_n|)r^2, \\
&\leq (1+|b_1|)r + \frac{1-\alpha}{(1+\beta)(2-\alpha)}\sum_{n=2}^{\infty}(1-\beta+n\beta)(\frac{n-\alpha}{1-\alpha}|a_n| + \frac{n+\alpha}{1-\alpha}|b_n|)r^2, \\
&\leq (1+|b_1|)r + \frac{1-\alpha}{(1+\beta)(2-\alpha)}\left(1 - \frac{1+\alpha}{1-\alpha}|b_1|\right)r^2, \qquad |z| = r < 1,
\end{aligned}$$

which proves the assertion (9) of Theorem 3.1. Similarly, the assertion (10) of Theorem 3.1 holds.

Further, we obtain the following and omit the proofs:

Theorem 3.2. *Let the function f defined by (2) be in the class $\mathcal{T}_{\mathcal{H}}(\alpha, \beta, n)$, $|z| = r < 1$, then*

$$|f'(z)| \leq 1 + |b_1| + 2\left(\frac{1-\alpha}{(1+\beta)(2-\alpha)} - \frac{1+\alpha}{(1+\beta)(2-\alpha)}\right)r \tag{11}$$

and

$$|f(z)| \geq 1 - |b_1| - 2\left(\frac{1-\alpha}{(1+\beta)(2-\alpha)} - \frac{1+\alpha}{(1+\beta)(2-\alpha)}\right)r. \tag{12}$$

The results are sharp.

4. Convolution Properties

Let the function h_m defined by

$$h_m(z) = z - \sum_{n=2}^{\infty}|a_{n,m}|z^n + \sum_{n=1}^{\infty}|b_{n,m}|\bar{z}^n, \tag{13}$$

with $m = (1,2)$ and be in the class $\mathcal{T}_{\mathcal{H}}(\alpha,\beta,n)$, we denote by $(h_1 * h_2)(z)$ the convolution (or Hadamard Product) of the function $h_1(z)$ and $h_2(z)$, that is,

$$(h_1 * h_2)(z) := z - \sum_{n=2}^{\infty} |a_{n,1}||a_{n,2}|z^n + \sum_{n=1}^{\infty} |b_{n,1}||b_{n,2}|\bar{z}^n. \tag{14}$$

We first show that the class $\mathcal{T}_{\mathcal{H}}(\alpha,\beta,n)$ is closed under convolution.

Theorem 4.1 *For $0 \le \gamma \le \alpha < 1$, let the functions $h_1 \in \mathcal{T}_{\mathcal{H}}(\alpha,\beta,n)$ and $h_2 \in \mathcal{T}_{\mathcal{H}}(\gamma,\beta,n)$. Then*

$$(h_1 * h_2)(z) \in \mathcal{T}_{\mathcal{H}}(\alpha,\beta,n) \in \mathcal{T}_{\mathcal{H}}(\gamma,\beta,n).$$

Proof.
Let $h_1 = z - \sum_{n=2}^{\infty} |a_{n,1}|z^n + \sum_{n=1}^{\infty} |b_{n,1}|\bar{z}^n$ be in $\mathcal{T}_{\mathcal{H}}(\alpha,\beta,n)$ and $h_2 = z - \sum_{n=2}^{\infty} |a_{n,2}|z^n + \sum_{n=1}^{\infty} |b_{n,2}|\bar{z}^n$ be in $\mathcal{T}_{\mathcal{H}}(\gamma,\beta,n)$. Then the convolution $h_1 * h_2$ is given by (14). We wish to show that the coefficients of $h_1 * h_2$ satisfy the required condition given in (8). For $h_2 \in \mathcal{T}_{\mathcal{H}}(\gamma,\beta,n)$, we note that $|a_{n,2}| < 1$ and $|b_{n,2}| < 1$. Now for the convolution functions $h_1 * h_2$ we obtain

$$\begin{aligned}
& \sum_{n=2}^{\infty}(1-\beta+n\beta)\frac{(n-\gamma)}{1-\gamma}|a_{n,1}||a_{n,2}| + \sum_{n=1}^{\infty}(1-\beta+n\beta)\frac{(n+\gamma)}{1-\gamma}|b_{n,1}||b_{n,2}| \\
\le\ & \sum_{n=2}^{\infty}(1-\beta+n\beta)\frac{(n-\gamma)}{1-\gamma}|a_{n,1}| + \sum_{n=1}^{\infty}(1-\beta+n\beta)\frac{(n+\gamma)}{1-\gamma}|b_{n,1}| \\
\le\ & \sum_{n=2}^{\infty}(1-\beta+n\beta)\frac{(n-\alpha)}{1-\alpha}|a_{n,1}| + \sum_{n=1}^{\infty}(1-\beta+n\beta)\frac{(n+\alpha)}{1-\alpha}|b_{n,1}| \le 1
\end{aligned}$$

since $0 \le \gamma \le \alpha < 1$ and $h_1 \in \mathcal{T}_{\mathcal{H}}(\alpha,\beta,n)$. Thus, $h_1 * h_2 \in \mathcal{T}_{\mathcal{H}}(\alpha,\beta,n) \subset \mathcal{T}_{\mathcal{H}}(\gamma,\beta,n)$.

Next we show that $\mathcal{T}_{\mathcal{H}}(\alpha,\beta,n)$ is closed under convex combinations of its members.

Theorem 4.2 *The class $\mathcal{T}_{\mathcal{H}}(\alpha,\beta,n)$ is closed under convex combination.*
Proof.

For $i = 1, 2, 3 \cdots$, let $h_i \in \mathcal{T}_{\mathcal{H}}(\alpha, \beta, n)$, where h_i is given by $h_i(z) = z - \sum_{n=2}^{\infty} |a_{n,i}| z^n + \sum_{n=1}^{\infty} |b_{n,i}| \bar{z}^n$. Then by (8)

$$\sum_{n=1}^{\infty} (1 - \beta + n\beta)[\frac{n-\alpha}{1-\alpha}|a_{n,i}| + \frac{n+\alpha}{1-\alpha}|b_{n,i}|] \leq 2. \tag{15}$$

For $\sum_{i=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of h_i shall be written as

$$\sum_{i=1}^{\infty} t_i h_i(z) = z - \sum_{n=2}^{\infty} (\sum_{i=1}^{\infty} t_i a_{n,i}) z^n + \sum_{n=1}^{\infty} (\sum_{i=1}^{\infty} t_i b_{n,i}) \bar{z}^n. \tag{16}$$

Then by (15),

$$\begin{aligned} & \sum_{n=1}^{\infty} (1 - \beta + n\beta)[\frac{n-\alpha}{1-\alpha} \sum_{i=1}^{\infty} t_i a_{n,i} + \frac{n+\alpha}{1-\alpha} \sum_{i=1}^{\infty} t_i b_{n,i}] \\ = \;& \sum_{i=1}^{\infty} t_i \left[\sum_{n=1}^{\infty} (1 - \beta + n\beta) \left(\frac{n-\alpha}{1-\alpha} a_{n,i} + \frac{n+\alpha}{1-\alpha} b_{n,i} \right) \right] \\ \leq \;& 2 \sum_{i=1}^{\infty} t_i = 2. \end{aligned}$$

Therefore $\sum_{i=1}^{\infty} t_i h_i(z) \in \mathcal{T}_{\mathcal{H}}(\alpha, \beta, n)$.

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