Tamsui Oxford Journal of Mathematical Sciences **24(3)** (2008) 215-223
Aletheia University

# Distortion Inequalities for the Dziok-Srivastava Linear Operator Involving the Generalized Hypergeometric Function*

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Received December 4, 2006, Accepted January 31, 2007.

**Abstract**

The object of the present paper is to discuss the coefficient estimates for multivalent functions belonging to the starlike and convex classes. Further, by using the Dziok-Srivastava linear operator, several distortion inequalities for these classes are also established.

**Keywords and Phrases:** *Analytic functions, Multivalent functions, Dziok-Srivastava linear operator, Starlike functions, Convex functions, Generalized hypergeometric functions, Distortion inequalities, Hadamard product (or Convolution).*

## 1. Introduction

Let $A_p$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p \in \mathbb{N} := \{1,2,3,\cdots\}) \tag{1.1}$$

which are *analytic* and *p-valent* in the open unit disk

$$U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}.$$

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* 2000 *Mathematics Subject Classification.* Primary 30C45; Secondary 33C20.

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A function $f \in A_p$ is said to be p-valent *starlike* function of order $\alpha$ $(0 \le \alpha < p)$, if and only if

$$\mathrm{Re}\left\{\frac{z f'(z)}{f(z)}\right\} > \alpha \qquad (z \in U). \tag{1.2}$$

We denote this class by $S_p^*(\alpha)$. On the other hand a function $f \in A_p$ is called p-valent *convex* function of order $\alpha$ $(0 \le \alpha < p)$, if and only if

$$\mathrm{Re}\left\{1 + \frac{z f''(z)}{f'(z)}\right\} > \alpha \qquad (z \in U). \tag{1.3}$$

We shall denote this class by $C_p(\alpha)$.

Further, we suppose that

$$h_p[(a_s);(b_t);z] = z^p \; {}_sF_t(a_1,\cdots,a_s;b_1,\cdots,b_t;z)$$

$$= z^p + \sum_{n=1}^{\infty} B_{p,n}^{(a_s),(b_t)} z^{p+n} \tag{1.4}$$

$$(s \le t+1;\, a_i \in \mathbb{R};\, b_j \in \mathbb{R} \setminus Z_0^-;\, Z_0^- = \{0,-1,-2,\cdots\};\, i = 1,\cdots,s;\, j = 1,\cdots,t;\, z \in U)$$

where ${}_sF_t$ is the well-known generalized hypergeometric function and

$$B_{p,n}^{(a_s),(b_t)} = \frac{\prod_{i=1}^{s}(a_i)_n}{\prod_{j=1}^{t}(b_j)_n \; (n)} . \tag{1.5}$$

Corresponding to the function $h_p[(a_s);(b_t);z]$, Dziok and Srivastava [1, p.3, Eq.(3)] introduced what is now well-known as the Dziok-Srivastava linear operator $H_p[(a_s);(b_t)]$ defined by the convolution

$$H_p[(a_s);(b_t)]f(z) = h_p[(a_s);(b_t);z] * f(z) \qquad (f \in A_p), \tag{1.6}$$

or, equivalently, by

$$H_p[(a_s);(b_t)]f(z) = z^p + \sum_{n=1}^{\infty} B_{p,n}^{(a_s),(b_t)} a_{p+n} z^{p+n} \qquad (z \in U), \qquad (1.7)$$

where the Hadamard product (convolution) of two analytic multivalent functions f(z) given by (1.1) and

$$g(z) = z^p + \sum_{n=1}^{\infty} b_{p+n} z^{p+n},$$

is denoted by $(f * g)(z)$ and is defined by

$$(f * g)(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} b_{p+n} z^{p+n}. \qquad (1.8)$$

## 2. Coefficient Estimates

Our result for the coefficient estimates of functions $f \in S_p^*(\alpha)$ is contained in

**Theorem 1.** *If* $f \in S_p^*(\alpha)$, *then*

$$|a_{p+n}| \leq \frac{1}{(n)!}\left[\prod_{j=2p}^{2p+n-1} (j - 2\alpha)\right] \qquad (n \geq 1). \qquad (2.1)$$

**Proof.** Let us define the function q (z) by

$$q(z) = \frac{1}{(p-\alpha)}\left[\frac{z f'(z)}{f(z)} - \alpha\right] \qquad (0 \leq \alpha < p) \qquad (2.2)$$

for $f \in S_p^*(\alpha)$. Then q (z) is analytic in U, q (0) = 1 and Re{q (z)} > 0 (z ∈ U). On substituting the value of f (z) from (1.1) in (2.2), we find that

$$q(z) = 1 + \sum_{n=1}^{\infty} q_n z^n. \qquad (2.3)$$

Obviously, $|q_n| \leq 2 \quad (n \geq 1)$. (2.4)

Since $z f'(z) - \alpha f(z) = (p-\alpha) q(z) f(z)$,

therefore

$$n a_{p+n} = (p-\alpha)[q_n + a_{p+1} q_{n-1} + a_{p+2} q_{n-2} + \ldots + a_{p+n-1} q_1] \tag{2.5}$$

Now we apply the principle of mathematical induction to prove our result (2.1).

If n = 1 in (2.5), then $a_{p+1} = (p-\alpha) q_1$. Now using (2.4), we find that

$$|a_{p+1}| \le 2(p-\alpha). \tag{2.6}$$

Thus the coefficient estimate (2.1) holds true for n = 1. Next, suppose that the coefficient estimate (2.1) is true for n = k (a fixed positive integer) i.e.

$$|a_{p+k}| \le \frac{1}{(k)!}\left[\prod_{j=2p}^{2p+k-1}(j-2\alpha)\right]. \tag{2.7}$$

Taking n = k+1 in (2.5) we get

$$(k+1) a_{p+k+1} = (p-\alpha)[q_{k+1} + a_{p+1} q_k + a_{p+2} q_{k-1} + \ldots + a_{p+k} q_1],$$

$$\Rightarrow (k+1)|a_{p+k+1}|$$

$$\le (2p-2\alpha)[1+|a_{p+1}|+|a_{p+2}|+\ldots+|a_{p+k}|] \qquad \{\text{by } (2.4)\}$$

$$\le (2p-2\alpha)\Bigg[1+(2p-2\alpha)+\frac{1}{2!}\{(2p-2\alpha)(2p-2\alpha+1)\}$$

$$+\ldots+\frac{1}{(k)!}\left\{\prod_{j=2p}^{2p+k-1}(j-2\alpha)\right\}\Bigg] \qquad \{\text{by } (2.1)\}$$

$$= (2p-2\alpha)\left[\frac{1}{(k-1)!}\{(2p-2\alpha+1)\ldots(2p-2\alpha+k-1)\}+\frac{1}{(k)!}\left\{\prod_{j=2p}^{2p+k-1}(j-2\alpha)\right\}\right]$$

$$= \frac{1}{(k)!}\left[\prod_{j=2p}^{2p+k}(j-2\alpha)\right].$$

Hence, the coefficient estimate (2.1) holds true for n = k +1 also. Thus by the principle of mathematical induction the result (2.1) for coefficient estimate is true for all n ∈ N. This completes the proof of the Theorem 1.

For the functions f(z) belonging to the class $C_p(\alpha)$, we similarly have

**Theorem 2.** *Let $f \in C_p(\alpha)$, then*

$$|a_{p+n}| \le \frac{p}{(p+n)\,n!}\left[\prod_{j=2p}^{2p+n-1}(j-2\alpha)\right] \qquad (n \ge 1). \tag{2.8}$$

By taking p = 1 in Theorems 1 and 2 we get the well-known results due to Robertson [3].

## 3. Distortion Inequalities for Starlike Functions and Convex Functions

Our first distortion inequality is contained in

**Theorem 3.** *If a function f(z) given by (1.1) belongs to the class* $S_p^*(\alpha)$, *then*

$$|H_p[(a_s);(b_t)]f(z)|$$
$$\le M_p[k,(a_s),(b_t),\alpha;|z|] + B_{p,k-1}^{(a_s),(b_t)}\frac{(2p-2\alpha)_{k-1}}{(k-1)!}|z|^{p+k-1}\,.$$
$${}_{s+2}F_{t+2}\left[\begin{matrix} k+(a_s)-1, k+2p-2\alpha-1, 1\,; \\ k+(b_t)-1, \quad k \;, k; \end{matrix}\; |z|\right] \tag{3.1}$$

*where* $k \in N \setminus \{1,2\}$, $a_i > 0$, $b_j > 0$, $i = 1,2,\cdots,s$, $j = 1,2,\cdots,t$,

$$M_p[k,(a_s),(b_t),\alpha;|z|] = |z|^p + \sum_{n=1}^{k-2} B_{p,n}^{(a_s),(b_t)}\frac{(2p-2\alpha)_n}{(n)!}\;|z|^{p+n} \tag{3.2}$$

*and* $B_{p,n}^{(a_s),(b_t)}$ *is as defined in (1.5).*

**Proof.** We begin by noting that

$$|a_{p+n}| \le \frac{\prod_{j=2p}^{2p+n-1}(j-2\alpha)}{(n)!} = \frac{(2p-2\alpha)_n}{(n)!} \tag{3.3}$$

By virtue of (1.5), (1.7) and (3.3), we have

$$|H_p[(a_s);(b_t)]f(z)|$$

$$\le |z|^p + \sum_{n=1}^{\infty} B_{p,n}^{(a_s),(b_t)} |a_{p+n}||z|^{p+n}$$

$$\le |z|^p + \sum_{n=1}^{\infty} \frac{\prod_{i=1}^{s}(a_i)_n (2p-2\alpha)_n}{\prod_{j=1}^{t}(b_j)_n \{n!\}^2} |z|^{p+n}$$

$$= \left\{ |z|^p + \sum_{n=1}^{k-2} \frac{\prod_{i=1}^{s}(a_i)_n (2p-2\alpha)_n}{\prod_{j=1}^{t}(b_j)_n \{(n)!\}^2} |z|^{p+n} \right\} + \sum_{n=k-1}^{\infty} \frac{\prod_{i=1}^{s}(a_i)_n (2p-2\alpha)_n}{\prod_{j=1}^{t}(b_j)_n \{(n)!\}^2} |z|^{p+n}$$

$$= M_p[k,(a_s),(b_t),\alpha;|z|] + \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{s}(a_i)_{k+n-1} (2p-2\alpha)_{k+n-1}}{\prod_{j=1}^{t}(b_j)_{k+n-1} \{(k+n-1)!\}^2} |z|^{p+n+k-1} \tag{3.4}$$

Since $(a)_{k+n-1} = (a)_{k-1}(k+a-1)_n$ and $(k+n-1)! = (k-1)!\,(k)_n$ therefore (3.4) reduces to

$$|H_p[(a_s);(b_t)]f(z)|$$

$$\le M_p[k,(a_s),(b_t),\alpha;|z|] + B_{p,k-1}^{(a_s),(b_t)} \frac{(2p-2\alpha)_{k-1}}{(k-1)!} |z|^{p+k-1}.$$

$$\sum_{n=0}^{\infty} \frac{\prod_{i=1}^{s} (k+a_i-1)_n (k+2p-2\alpha-1)_n}{\prod_{j=1}^{t} (k+b_j-1)_n \{(k)_n\}^2} |z|^n$$

$$= M_p[k,(a_s),(b_t),\alpha;|z|]$$

$$+ B_{p,k-1}^{(a_s),(b_t)} \frac{(2p-2\alpha)_{k-1}}{(k-1)!} |z|^{p+k-1} \; {}_{s+2}F_{t+2}\left[\begin{matrix} k+(a_s)-1, k+2p-2\alpha-1, 1\,; \\ k+(b_t)-1, \quad k \;,\; k\,; \end{matrix} |z|\right]$$

which completes the proof of Theorem 3.

**Corollary 1.** *If a function f(z) given by (1.1) belongs to the class* $S_p^*\left(\frac{2p-1}{2}\right)$, *then*

$$|H_p[(a_s);(b_t)]f(z)|$$

$$\le |z|^p + \sum_{n=1}^{k-2} B_{p,n}^{(a_s),(b_t)} |z|^{p+n} + B_{p,k-1}^{(a_s),(b_t)} |z|^{p+k-1} \; {}_{s+1}F_{t+1}\left[\begin{matrix} k+(a_s)-1, 1 \;; \\ k+(b_t)-1, k \;; \end{matrix} |z|\right]. \quad (3.5)$$

Further taking $\alpha = 0$ in Theorem 3, we have

**Corollary 2.** *If a function f(z) given by (1.1) belongs to the class* $S_p^*$, *then*

$$|H_p[(a_s);(b_t)]f(z)|$$

$$\le |z|^p + \sum_{n=1}^{k-2} B_{p,n}^{(a_s),(b_t)} \frac{(2p)_n}{(n)!} |z|^{p+n}$$

$$+ B_{p,k-1}^{(a_s),(b_t)} \frac{(2p)_{k-1}}{(k-1)!} |z|^{p+k-1} \; {}_{s+2}F_{t+2}\left[\begin{matrix} k+(a_s)-1, k+2p-1, 1 \;; \\ k+(b_t)-1, \quad k \quad , k\,; \end{matrix} |z|\right]. \quad (3.6)$$

**Theorem 4.** *If a function f(z) given by (1.1) belongs to the class $C_p(\alpha)$, then*

$$|H_p[(a_s);(b_t)]f(z)|$$

$$\leq |z|^p + \sum_{n=1}^{k-2} B_{p,n}^{(a_s),(b_t)} \frac{p(2p-2\alpha)_n}{(p+n)(n)!} |z|^{p+n}$$

$$+ B_{p,k-1}^{(a_s),(b_t)} \frac{p(2p-2\alpha)_{k-1}}{(k+p-1)(k-1)!} |z|^{p+k-1} \quad {}_{s+3}F_{t+3}\left[\begin{matrix} k+(a_s)-1, k+2p-2\alpha-1, k+p-1, 1 \; ; \\ k+(b_t)-1, \quad k \quad , \quad k+p \quad , \; k \; ; \end{matrix} |z|\right] \tag{3.7}$$

*where* $k \in N \setminus \{1,2\}, a_i > 0, b_j > 0, i = 1,2,\cdots,s, j = 1,2,\cdots,t,$ *and* $B_{p,n}^{(a_s),(b_t)}$ *is as defined in (1.5).*

**Proof.** Using the fact that

$$|a_{p+n}| \leq \frac{p\left[\prod_{j=2p}^{2p+n-1}(j-2\alpha)\right]}{(p+n)n!} = \frac{p[(2p-2\alpha)_n]}{(p+n)(n)!} \tag{3.8}$$

and proceeding on the similar lines as that in Theorem 3, we readily arrive at the inequality (3.7).

**Corollary 3.** *If a function f(z) given by (1.1) belongs to the class* $C_p\left(\frac{2p-1}{2}\right)$, *then*

$$|H_p[(a_s);(b_t)]f(z)|$$

$$\leq |z|^p + \sum_{n=1}^{k-2} B_{p,n}^{(a_s),(b_t)} \frac{p}{(p+n)} |z|^{p+n}$$

$$+ B_{p,k-1}^{(a_s),(b_t)} \frac{p}{k+p-1} |z|^{p+k-1} \; {}_{s+2}F_{t+2}\left[\begin{matrix} k+(a_s)-1, k+p-1, 1 \; ; \\ k+(b_t)-1, k+p, k \quad ; \end{matrix} |z|\right] \tag{3.9}$$

Further, taking $\alpha = 0$ in Theorem 4, we have

**Corollary 4.** *If a function f(z) given by (1.1) belongs to the class* $C_p$, *then*

$$|H_p[(a_s);(b_t)]f(z)|$$

$$\leq |z|^p + \sum_{n=1}^{k-2} B_{p,n}^{(a_s),(b_t)} \frac{p(2p)_n}{(p+n)(n)!}|z|^{p+n}$$

$$+ B_{p,k-1}^{(a_s),(b_t)} \frac{p(2p)_{k-1}}{(k+p-1)(k-1)!}|z|^{p+k-1} \quad {}_{s+3}F_{t+3}\left[\begin{matrix} k+(a_s)-1, k+2p-1, k+p-1, 1 \;; \\ k+(b_t)-1, \quad k \quad , \quad k+p \quad , k \;; \end{matrix} |z|\right] . \tag{3.10}$$

Setting $p = s = 1$, $t = 0$, $a_1 = \lambda+1$ in Theorems 3 and 4, the results obtained by Owa and Srivastava become particular cases of our investigation [2, pp. 241 and 244, Theorems 1 and 2].

# Acknowledgements

The authors are thankful to Professor H. M. Srivastava (University of Victoria, Canada) for his kind help and useful suggestions during the preparation of this work.

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