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ON MAPS THAT PRESERVE $*$-PRODUCTS OF OPERATORS IN $\mathcal{B}(\mathcal{H})$

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Abstract. Let $\mathcal{B}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators on Hilbert space $\mathcal{H}$. In this paper, we introduce three concept of $*$-products on $\mathcal{B}(\mathcal{H})$, and present a form for a unital surjective bounded linear map $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, which preserves these $*$-products in both directions.

1. Introduction and Preliminaries

Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are linear spaces and $\varphi : X \to Y$ is a map. We say that φ is a preserving map in both directions, whenever

$$x \in \mathcal{X} \text{ has the property } p \Leftrightarrow \varphi(x) \in \mathcal{Y} \text{ has the property } p.$$

In mathematics, the linear preserving problems usually define on matrix spaces or operator spaces and many of mathematicians study on linear maps that act on those spaces, where preserve some properties such as Birkhoff orthegonality, unitary operators, spectral radius, quasi-isometry,..., and they try to present special form for such linear maps, which mentioned. For more details one can see, [2, 5, 9, 10].

Suppose that $\mathcal{H}$ is a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ the C^*-algebra of all bounded linear operators on $\mathcal{H}$. In this paper, first we define three concept of $*$-products on $\mathcal{B}(\mathcal{H})$ and we characterize the surjective bounded linear maps $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, which preserve

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these $*$-products in both directions. For each self-adjoint operator $S \in \mathcal{B}(\mathcal{H})$ the operator $\exp(itS) \in \mathcal{B}(\mathcal{H})$ plays substantial role in the proof of the main theorems.

Let $K \in \mathcal{B}(\mathcal{H})$ be a nonzero positive invertible element. If we define $\langle x, y\rangle_K = \langle Kx, y\rangle$ for each $x, y \in \mathcal{H}$, then $\langle \cdot, \cdot\rangle_K$ is a inner product on $\mathcal{H}$ and $\mathcal{H}_K = (\mathcal{H}, \langle \cdot, \cdot\rangle_K)$ is a Hilbert space and $T^\sharp = K^{-1}T^*K$ is the K-adjoint of T with respect to the inner product $\langle \cdot, \cdot\rangle_K$, where T^* is the adjoint of T with respect to the inner product $\langle \cdot, \cdot\rangle$ (that is $\langle Tx, y\rangle = \langle x, T^*y\rangle \ \ x, y \in \mathcal{H}$), see [3].

Recall that $\sharp$ is an involution on $\mathcal{B}(\mathcal{H})$ and the set of all bounded linear operators on $\mathcal{H}$ with respect to inner product $\langle ., .\rangle_K$ is the same as $\mathcal{B}(\mathcal{H}, \langle ., .\rangle)$

Definition 1.1. Let $S \in \mathcal{B}(\mathcal{H})$. Then S is called K-self-adjoint, K-unitary, if $K^{-1}S^*K = S$ and $TK^{-1}T^*K = K^{-1}T^*KT = I$, respectively.

Recall that a C^*-algebra $\mathcal{A}$ is said to be prime if for each $a, b \in \mathcal{A}$, $a\mathcal{A}b = \{0\}$ implies $a = 0$ or $b = 0$.

Lemma 1.2. [9, P. 3] *The C^*-algebra $\mathcal{B}(\mathcal{H})$ is prime.*

Definition 1.3. Let $\mathcal{A}$ and $\mathcal{B}$ be two $*$-algebras. A linear map $\varphi : \mathcal{A} \to \mathcal{B}$ is called a $*$-Jordan homomorphism if $\varphi(a^2) = \varphi(a)^2$ and $\varphi(a^*) = \varphi(a)^*$ for every $a \in \mathcal{A}$.

Theorem 3.1 of [7] ensures that a $*$-Jordan homomorphism onto a prime C^*-algebra is either a $*$-homomorphism or a $*$-anti-homomorphism.

2. Linear maps that preserve $*$-product

In this section, we characterize the unital linear maps from $\mathcal{B}(\mathcal{H})$ onto itself that preserve $*$-product, 3-order $*$-product, and inverse $*$-product in both directions.

Definition 2.1. Let $T, S \in \mathcal{B}(\mathcal{H})$. We say that T and S are $*$-product and denoted by $T * S$, whenever TS^* is a projection. A map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is said to be a $*$-product preserving in both directions, when TS^* is a projection if and only if $\varphi(T)\varphi(S)^*$ is a projection.

Example 2.2. Let $T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in \mathcal{M}_2(\mathbb{C})$ and $*$ is the conjugate transpose. Then $T^* = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Thus

$$TT^* = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad (TT^*)^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore T is a $*$-product (with itself), since TT^* is projection.

Lemma 2.3. [11, Theorem 2.3.3] *Let $T \in \mathcal{B}(\mathcal{H})$. The following statements are equivalent:*
(i) TT^* is a projection
(ii) $TT^*T = T$
(iii) $T^*TT^* = T^*$.

To achieve our next result, we utilize the strategy of [9].

Theorem 2.4. *Let $T \in \mathcal{B}(\mathcal{H})$ and $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a unital surjective bounded linear map. If φ preserves $*$-product in both directions, then there are unitary operators $U, V \in \mathcal{B}(\mathcal{H})$ such that*

$$\varphi(T) = UTV \quad \text{or} \quad \varphi(T) = UT^{tr}V$$

where T^{tr} is the transpose of T with respect to an arbitrary but fixed orthonormal basis of $\mathcal{H}$.

Proof. Pick a self-adjoint operator S in $\mathcal{B}(\mathcal{H})$. Then $\exp(itS)^* = \exp(-itS)$. Thus

$$\exp(itS)\exp(itS)^* = I,$$

that is $\exp(itS)$ is a $*$-product with itself. Therefore $\varphi(exp(itS)) * \varphi(exp(itS))$ is a projection, since φ preserves $*$-product. Lemma 2.3 implies that

$$\begin{aligned}
\varphi(\exp(itS)) &= \varphi(\exp(itS))\varphi(\exp(itS))^*\varphi(\exp(itS)) \\
&= \varphi(I + itS + \frac{(it)^2}{2!}S^2 + \cdots)\varphi(I + itS + \frac{(it)^2}{2!}S^2 + \cdots)^* \\
&\quad \varphi(I + itS + \frac{(it)^2}{2!}S^2 + \cdots) \\
&= I + it(2\varphi(S) - \varphi(S)^*) + t^2(\varphi(S)^*\varphi(S) + \varphi(S)\varphi(S)^* \\
&- \frac{1}{2}\varphi(S^2)^* - \varphi(S^2) - \varphi(S)^2) \cdots .
\end{aligned}$$

Hence

$$
\begin{aligned}
I + it\varphi(S) - \frac{t^2}{2}\varphi(S^2) + \cdots &= I + it(2\varphi(S) - \varphi(S)^*) \\
&+ t^2(\varphi(S)^*\varphi(S) + \varphi(S)\varphi(S)^* \\
&- \frac{1}{2}\varphi(S^2)^* - \varphi(S^2) - \varphi(S)^2) \cdots .
\end{aligned}
$$

It follows that

$$\varphi(S) = \varphi(S)^* \tag{2.1}$$

and

$$\varphi(S)^*\varphi(S) + \varphi(S)\varphi(S)^* - \frac{1}{2}\varphi(S^2)^* - \frac{1}{2}\varphi(S^2) - \varphi(S)^2 = 0 \tag{2.2}$$

for every self-adjoint operator $S \in \mathcal{B}(\mathcal{H})$. By (2.1), (2.2) and similar to the proof of [6, Theorem 3.3.], for every $T \in \mathcal{B}(\mathcal{H})$ we have

(i) $\varphi(T^*) = \varphi(T)^*$,

(ii) $\varphi(T^2) = \varphi(T)^2$.

Therefore φ is a $*$-Jordan homomorphism. It is known that every $*$-Jordan homomorphism on a prime algebra is a $*$-homomorphism or a $*$-anti-homomorphism. Since $\mathcal{B}(\mathcal{H})$ is a prime algebra, φ is a $*$-homomorphism or a $*$-anti-homomorphism. Now we show that φ is injective. Let $S \in \mathcal{B}(\mathcal{H})$ be self-adjoint and $\varphi(S) = 0$. Then $\varphi(S + I) = I$ and $\varphi(S - I) = -I$, since $\varphi(I) = I$. Clearly $I * I$ and $-I * -I$ are projections. On the other hand φ preserves $*$-product in both directions, so $(S + I) * (S + I)$ and $(S - I) * (S - I)$ are projections. By Lemma 2.3 (ii), we get $S - S^2 = 0$ and $S + S^2 = 0$. Thus $S = 0$. Let $T \in \mathcal{B}(\mathcal{H})$ be an arbitrary element and $\varphi(T) = 0$. There exist self-adjoint operators $S_1, S_2 \in \mathcal{B}(\mathcal{H})$ such that $T = S_1 + iS_2$ and

$$\varphi(S_1) + i\varphi(S_2) = \varphi(T) = 0 = \varphi(T)^* = \varphi(S_1) - i\varphi(S_2).$$

So $\varphi(S_1) = 0$ and $\varphi(S_2) = 0$. Therefore $S_1 = S_2 = 0$, and also $T = 0$, which implies that φ is injective. Then it is a $*$-automorphism or a $*$-anti-automorphism. □

Corollary 2.5. *Let $T \in \mathcal{B}(\mathcal{H})$ and $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a unital surjective bounded linear map. If φ preserves $\sharp$-product in both directions, then there are K-unitary operators $U, V \in \mathcal{B}(\mathcal{H})$ such that*

$$\varphi(T) = UTV \quad \text{or} \quad \varphi(T) = UT^{tr}V$$

where T^{tr} is the transpose of T with respect to an arbitrary but fixed orthonormal basis of $\mathcal{H}$.

Definition 2.6. Let $T, S \in \mathcal{B}(\mathcal{H})$. Then T and S are 3-order $*$-product denoted by $T *^3 S$, whenever $T^3S^{*3} = TS^*$.

Example 2.7. Let $T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \in \mathcal{M}_2(\mathbb{C})$ and $*$ is the conjugate transpose. Then

$$T^* = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, T^{*3} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, T^3 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

A straightforward computation shows that $T^3T^{*3} = TT^*$. Hence T is a 3-order $*$-product with itself.

Theorem 2.8. *Let $T \in \mathcal{B}(\mathcal{H})$ and $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a unital surjective bounded linear map. If φ preserves 3-order $*$-product in both directions, then there are unitary operators $U, V \in \mathcal{B}(\mathcal{H})$ such that*

$$\varphi(T) = UTV \quad \text{or} \quad \varphi(T) = UT^{tr}V$$

where T^{tr} is the transpose of T with respect to an arbitrary but fixed orthonormal basis of $\mathcal{H}$.

Proof. We utilize the strategy of previous theorem. Since $\exp(itS)^* = \exp(-itS)$, so

$$\exp(itS)^3\exp(itS)^{*3} = I = \exp(itS)\exp(itS)^*,$$

that is $\exp(itS)$ is a 3-order $*$-product with itself, for all $t \in \mathbb{R}$. On the other hand φ preserves 3-order $*$-product, thus $\varphi(exp(itS))^3\varphi(exp(itS))^{*3} = \varphi(exp(itS))\varphi(exp(itS))^*$. Lemma 2.3 implies that

$$I + 3it(\varphi(S) - \varphi(S)^*) + t^2(-\frac{3}{2}(\varphi(S^2)^* + \varphi(S^2)) - 3(\varphi(S)^{*2} + \varphi(S)^2) + 9\varphi(S)\varphi(S)^*) + ... =$$
$$I + it(\varphi(S) - \varphi(S)^*) + t^2(-\frac{1}{2}(\varphi(S^2)^* + \varphi(S^2)) + \varphi(S)\varphi(S)^*)) + ...,$$

so $\varphi(S) = \varphi(S)^*$ and consequently $\varphi(S^2) = \varphi(S)^2$. Thus for every $T \in \mathcal{B}(\mathcal{H})$, we have

(i) $\varphi(T^*) = \varphi(T)^*$,

(ii) $\varphi(T^2) = \varphi(T)^2$.

Therefore φ is a $*$-Jordan homomorphism. In continue, we prove that φ is an injective operator. Let $S \in \mathcal{B}(\mathcal{H})$ be self-adjoint and $\varphi(S) = 0$. Clearly

$$\varphi(S - I)^3\varphi(S - I)^{*3} = \varphi(S - I)\varphi(S - I)^*,$$

$$\varphi(S+I)^3\varphi(S+I)^{*3} = \varphi(S+I)\varphi(S+I)^*.$$

Since φ preserves 3-order $*$-product in both directions, we reach $2S^6 + 30S^4 + 28S^2 = 0$. Then $S = \pm 14i, \pm i, 0$. Therefore $S = 0$, since S is self-adjoint. The rest is similar to the proof of Theorem 2.4. □

Definition 2.9. Let $T, S \in \mathcal{B}(\mathcal{H})$. Then T and S are inverse $*$-product denoted by $T *^{-1} S$, whenever $TS^* + S^*T = 2I$.

Example 2.10. Let $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathcal{B}(\mathcal{H})$, then $T^* = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Clearly $TT^* + T^*T = 2I$, so T is a inverse $*$-product.

Theorem 2.11. *Let $T \in \mathcal{B}(\mathcal{H})$ and $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a unital surjective bounded linear map. If φ preserves inverse $*$-product in both directions, then there are unitary operators $U, V \in \mathcal{B}(\mathcal{H})$ such that*

$$\varphi(T) = UTV \quad \text{or} \quad \varphi(T) = UT^{tr}V$$

where T^{tr} is the transpose of T with respect to an arbitrary but fixed orthonormal basis of $\mathcal{H}$.

Proof. We know that $\exp(itS)\exp(itS)^* + \exp(itS)^*\exp(itS) = 2I$. Since φ preserves inverse $*$-product so $\varphi(\exp(itS))\varphi(\exp(itS))^* + \varphi(\exp(itS))^*\varphi(\exp(itS)) = 2I$. Thus

$$(I + it\varphi(S) - \frac{(t)^2}{2!}\varphi(S^2) + \cdots)(I - it\varphi(S)^* - \frac{(t)^2}{2!}\varphi(S^2)^* + \cdots) +$$
$$(I - it\varphi(S)^* - \frac{(t)^2}{2!}\varphi(S^2)^* + \cdots)(I + it\varphi(S) - \frac{(t)^2}{2!}\varphi(S^2) + \cdots) = 2I.$$

Therefore, we get

$$2I + it(-2\varphi(S)^* + 2\varphi(S)) + t^2(-\varphi(S^2)^* + \varphi(S)\varphi(S)^* - \varphi(S^2) + \varphi(S)^*\varphi(S)) + \cdots = 2I.$$

From the equation above, we reach $\varphi(S) = \varphi(S)^*$ and $\varphi(S^2) = \varphi(S)^2$. Thus for every $T \in \mathcal{B}(\mathcal{H})$, we have

(i) $\varphi(T^*) = \varphi(T)^*$,

(ii) $\varphi(T^2) = \varphi(T)^2$.

Therefore φ is a $*$-Jordan homomorphism. Let $S \in \mathcal{B}(\mathcal{H})$ be self-adjoint and $\varphi(S) = 0$. Thus

$$\varphi(I-S)\varphi(I-S)^* + \varphi(I-S)^*\varphi(I-S) = 2I,$$

$$\varphi(I+S)\varphi(I+S)^* + \varphi(I+S)^*\varphi(I+S) = 2I.$$

Since φ preserves inverse $*$-product in both directions, we reach $(I-S)(I-S)^*+(I-S)^*(I-S) = 2I$ and $(I+S)(I+S)^* + (I+S)^*(I+S) = 2I$. Therefore $S^2 - 4S = 0,\ S^2 + 4S = 0$. Consequently $S = 0$. □

By Theorem 2.8 and Theorem 2.11, we conclude the following corollaries.

Corollary 2.12. *Let $T \in \mathcal{B}(\mathcal{H})$ and $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a unital surjective bounded linear map. If φ preserves 3-order $\sharp$-product in both directions, then there are K-unitary operators $U, V \in \mathcal{B}(\mathcal{H})$ such that*

$$\varphi(T) = UTV \quad \text{or} \quad \varphi(T) = UT^{tr}V$$

where T^{tr} is the transpose of T with respect to an arbitrary but fixed orthonormal basis of $\mathcal{H}$.

Corollary 2.13. *Let $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a unital surjective bounded linear map. If φ preserves inverse $\sharp$-product in both directions, then there are K-unitary operators $U, V \in \mathcal{B}(\mathcal{H})$ such that*

$$\varphi(T) = UTV \quad \text{or} \quad \varphi(T) = UT^{tr}V$$

where T^{tr} is the transpose of T with respect to an arbitrary but fixed orthonormal basis of $\mathcal{H}$.

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