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Stability Of A Two -Variable Pexiderized Additive Functional Equation In Intuitionistic Fuzzy Banach Spaces: A Fixed Point Approach

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Abstract

In this paper we show that Hyers-Ulam-Rassias stability holds for Pexiderized additive functional equations in the framework of intuitionistic fuzzy Banach spaces. We consider a two-variable pexiderized additive functional equation. The approach to the present stability problem is through fixed point method for which we apply the Banach contraction mapping principle in generalized metric spaces which is a structure where infinite distances are admissible.

Keywords : Hyers-Ulam stability; Pexider type functional equation; intuitionistic fuzzy normed spaces; alternative fixed point theorem.

1 Introduction

In Mathematics, functional equation are studied in various contexts. In particular the study on the stabilities of functional equation has a large literature. It started with the basic question raised by Ulam [20] which was partially answered by Hyers [9] and Raasias [16]. This type of stability is now known as Hyers-Ulam-Rassias stability or in short H-U-R stability.

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Our background of work is intuitionistic fuzzy metric space [15] . As is well known, the fuzzy concept introduced in 1965 by Zadeh [22] created a new trend in almost all branches of pure and applied Mathematics. In particular fuzzy mathematics was introduced in linear algebra and functional analysis which has led to the definition of fuzzy Banach spaces. The concept of fuzzy set has been further extended in many directions [4, 19] of which intuitionistic fuzzy set [1] is an instance where in addition to the degree of membership there is also a degree of non-membership. Fuzzy Banach spaces are also extended to intuitionistic fuzzy Banach spaces in works like [10, 11, 13, 14, 21].
We establish a stability result for a Pexiderized additive functional equation in intuitionistic fuzzy normed linear spaces [2] . The equation involves two variables. We apply an extension of the Banach contraction mapping principle in generalized metric spaces to our problem. Generalized metric space was defined [8] by modifying the metric into an extended real valued functions which can take up infinite values. An illustrative example is also discussed.

2 Mathematical Preliminaries:

Definition 2.1. [7] Consider the set L^* and the order relation $\leq_{L^*}$ defined by

$$Ł^* = \{ (x_1, x_2) : (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 + x_2 \leq 1 \},$$

$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1, x_2 \geq y_2, \forall (x_1, x_2), (y_1, y_2) \in L^*.$$

Then $(L^*, \leq_{L^*})$ is a complete lattice.
The elements $0_{L^*} = (0, 1)$ and $1_{L^*} = (1, 0)$ are its units.

Definition 2.2. [22] A fuzzy set A of a non-empty set X is characterized by a membership function μ_A which associates each point of X to a real number in the interval [0, 1]. With the value of $\mu_A(x)$ at x representing the grade of membership of x in A.

Definition 2.3. [1] Let E be any nonempty set. An intuitionistic fuzzy set A of E is an object of the form $A = \{(x, \mu_A(x), \nu_A(x)) : x \in E\}$, where the functions $\mu_A : E \to [0, 1]$ and $\nu_A : E \to [0, 1]$ denote the degree of membership and the degree of non-membership of the element $x \in E$ respectively and for every $x \in E$,
$0 \leq \mu_A(x) + \nu_A(x) \leq 1$.
For our notational purposes we denote an intuitionistic fuzzy set on X by a function $A_{\mu,\nu} =: X \to L^*$ given by $A_{\mu,\nu}(x) = (\mu_A(x), \nu_A(x))$ with $\mu_A, \nu_A : X \to [0,1]$ satisfying $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

Definition 2.4. [7] A triangular norm (t-norm) on L^* is a mapping
$\Gamma : (L^*)^2 \to L^*$ satisfying the following conditions :

(a) $(\forall x \in L^*)(\Gamma(x, 1_{L^*}) = x)$ (boundary condition),

(b) $(\forall (x, y) \in (L^*)^2)(\Gamma(x, y) = \Gamma(y, x))$ (commutativity),

(c) $(\forall (x, y, z) \in (L^*)^3)(\Gamma(x, \Gamma(y, z)) = \Gamma(\Gamma(x, y), z))$ (associativity),

(d) $(\forall (x, x', y, y') \in (L^*)^4)(x \leq_{L^*} x'$ and $y \leq_{L^*} y' \Rightarrow \Gamma(x, y) \leq_{L^*} \Gamma(x', y'))$ (monotonicity).

Definition 2.5. [7] A triangular conorm (t-conorm) on L^* is a mapping $S : (L^*)^2 \to L^*$ satisfying the following conditions :

(a) $(\forall x \in L^*)(S(x, 0_{L^*}) = x)$ (boundary condition),

(b) $(\forall (x, y) \in (L^*)^2)(S(x, y) = S(y, x))$ (commutativity),

(c) $(\forall (x, y, z) \in (L^*)^3)(S(x, S(y, z)) = S(S(x, y), z))$ (associativity),

(d) $(\forall (x, x', y, y') \in (L^*)^4)(x \leq_{L^*} x'$ and $y \leq_{L^*} y' \Rightarrow S(x, y) \leq_{L^*} S(x', y'))$ (monotonicity).
If Γ is continuous then Γ is said to be a continuous t-norm.

Definition 2.6. [7] A continuous t-norm Γ on L^* is said to be continuous t-representable if there exists a continuous t-norm $*$ and there exists a continuous t-conorm $\diamond$ on $[0, 1]$ such that for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in L^*$, $\Gamma(x, y) = (x_1 * y_1, x_2 \diamond y_2)$
We now define the iterated sequence Γ^n recursively by $\Gamma^1 = \Gamma$ and

$$\Gamma^n(x^{(1)}, x^{(2)}, \cdots, x^{(n+1)}) = \Gamma(\Gamma^{(n-1)}(x^{(1)}, x^{(2)} \cdots, x^{(n)}), x^{(n+1)}),$$

$\forall\, n \geq 2, x^{(i)} \in L^*$.

Intuitionistic fuzzy normed linear space was defined by Saadati [17]. Shakeri [18] has stated this definition in more compact form. We state the definition in the form used by Shakeri [18]

Definition 2.7. The triple $(X, P_{\mu,\nu}, T)$ is said to be an intuitionistic fuzzy normed space (briefly IFN-space) if X is a vector space, T is a continuous t-norm and $P_{\mu,\nu}$ is a mapping $X \times (0, \infty) \to L^*$ which is an intuitionistic fuzzy set satisfying the following conditions :
for all $x, y \in X$ and $t, s > 0$,

$(i)\ P_{\mu,\nu}(x, 0) = 0_{L^*}$;

$(ii)\ P_{\mu,\nu}(x, t) = 1_{L^*}$ if and only if $x = 0$;

$(iii)\ P_{\mu,\nu}(\alpha x, t) = P_{\mu,\nu}\left(x, \frac{t}{|\alpha|}\right)$ for all $\alpha \neq 0$;

$(iv)\ P_{\mu,\nu}(x + y, t + s) \geq_{L^*} \Gamma(P_{\mu,\nu}(x, t), P_{\mu,\nu}(y, s))$.

It can be noted that $P_{\mu,\nu}$ has the form
$P_{\mu,\nu}(x, t) = (\mu_x(t), \nu_x(t)) = (\mu(x, t), \nu(x, t))$ such that
$0 \leq \mu_x(t) + \nu_x(t) \leq 1$ for all $x \in X$ and $t > 0$. Then with μ and ν the above definition reduces to the more explicit form used in [17].

Example 2.8. Let $(X, \|.\|)$ be a normed linear space. Let

$$M(a, b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\})$$

for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$ and for $a, b \in [0, 1]$ Then $M(a, b)$ is continuous t-norm.

Definition 2.9. (1) A sequence $\{x_n\}$ in an IFN-space $(X, P_{\mu,\nu}, M)$ is called a Cauchy sequence if for any $\varepsilon > 0$ and $t > 0$, there exists $n_0 \in N$ such that

$$P_{\mu,\nu}(x_n - x_m, t) >_{L^*} (1 - \varepsilon, \varepsilon), \forall n, m \geq n_0$$

(2) The sequence $\{x_n\}$ is said to be convergent to a point $x \in X$ if

$$P_{\mu,\nu}(x_n - x, t) \to 1_{L^*} \text{ as } n \to \infty \text{ for every } t > 0.$$

(3) An IFN-space $(X, P_{\mu,\nu}, M)$ is said to be complete if every Cauchy sequence in X is convergent to a point $x \in X$. A complete intuitionistic fuzzy normed space is called an intuitionistic fuzzy Banach space.

We require the following generalized metric space and the fixed point result in generalized metric spaces to establish our result of stability in this paper.

Definition 2.10. Let X be a nonempty set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if d satisfies
(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.
Then (X, d) is called a generalized metric space or a g. m. s..

The essence of the above definition is that the distance function can now assume infinite value.

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Definition 2.11. [3]) Let (X, d) be a g. m. s., $\{x_n\}$ be a sequence in X, and $x \in X$. We say that $\{x_n\}$ is g. m. s. convergent to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$. We denote this by $x_n \to x$.

Definition 2.12. [3]) Let (X, d) be a g. m. s. and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is Cauchy sequence if and only if for each $\varepsilon > 0$, there exists a natural number N such that $d(x_n, x_m) < N$ for all $n > m > N$.

Definition 2.13. [3]) Let (X, d) be a g. m. s. Then (X, d) is called a complete g. m. s. if every g. m. s. Cauchy seuqence is g. m. s. convergent in X.

Theorem 2.14. *([12] and [5]) Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $0 < L < 1$, that is,*

$$d(Jx, Jy) \leq L\, d(x, y),$$

for all $x, y \in X$.
Then for each $x \in X$, either

$$d(J^n x, J^{n+1} x) = \infty, \ \forall\, n \geq 0$$

or,

$$d(J^n x, J^{n+1} x) < \infty \ \ \forall\, n \geq n_o$$

for some non-negative integers n_0. Moreover, if the second alternative holds then
(1) the sequence $\{J^n x\}$ converges to a fixed point $y^\star$ of J;
(2) $y^\star$ is the unique fixed point of J in the set

$$Y = \{y \in X : d(J^{n_0} x, y) < \infty\};$$

(3) $d(y, y^\star) \leq (\frac{1}{1-L})\, d(y, Jy)$ for all $y \in Y$.

The following is the definition of Pexiderized additive functional equation in two variables.

A mapping $f : R\times R \to R$ is said to be an additive form if $f(x, y) = ax+by$ for all $x, y, a, b \in R$.
If X and Y are assumed to be a real vector space and a Banach space respectively then for a mapping $f : X \times X \to Y$, consider two variables functional equation corresponding the Cauchy functional equation

$$f(x + y) = f(x) + f(y)$$

as

$$f(x+y, z+w) = f(x, z) + f(y, w) \quad \cdots \quad (1).$$

Any solution of (1) is termed as an additive mapping. Particularly, if $X = Y = R$, the additive form $f(x, y) = ax + by$ is a solution of (1). The form

$$f(x+y, z+w) = g(x, z) + h(y, w) \quad \cdots \quad (2)$$

is known as Pexiderized additive functional equation in two variables which is an extension of the above definition of additive functional equation [6].

3 The H-U-R Stability Result

Theorem 3.1. *Let X be a real linear space, $(Z, P'_{\mu,\nu}, \Gamma)$ be a IFN-space, $\phi : X \times X \times X \times X \to Z$ be a function such that*

$$P'_{\mu,\nu}(\phi(3x, 3y, 3z, 3w), t) \geq_{L^*} P'_{\mu,\nu}(\alpha\phi(x, y, z, w), t) \quad (3.1)$$

for some real α with $0 < \alpha < 3$, and

$$\lim_{n\to\infty} P'_{\mu,\nu}(\phi(3^n x, 3^n y, 3^n z, 3^n w), 3^n t) = 1_{L^*} \quad (3.2)$$

for all $x, y, z, w \in X$ and $t > 0$. Let $(Y, P_{\mu,\nu}, \Gamma)$ be a complete IFN-space. If $f, g, h : X \times X \to Y$ are mappings such that

$$P_{\mu,\nu}(f(x+y, z+w) - g(x, z) - h(y, w), t) \geq_{L^*} P'_{\mu,\nu}(\phi(x, y, z, w), t) \quad (3.3)$$

$(x, y, z, w \in X, t > 0)$.
Then there exists a unique additive mapping $A : X \times X \to Y$ define by $A(x, z) := \lim_{n\to\infty} \left(\frac{f(3^n x, 3^n z) - f(0,0)}{3^n} \right)$ for all $x, z \in X$ satisfying

$$P_{\mu,\nu}(f(x, z) - f(0, 0) - A(x, z), t) \geq_{L^*} M_1((x, z), t(3 - \alpha)) \quad (3.4)$$

where

$$M_1((x, z), t)$$
$$= \Gamma^7 \left\{ P'_{\mu,\nu}\left(\phi\left(\frac{-x}{2}, \frac{3x}{2}, \frac{-z}{2}, \frac{3z}{2}\right), \frac{t}{8}\right), P'_{\mu,\nu}\left(\phi\left(\frac{3x}{2}, \frac{-x}{2}, \frac{3z}{2}, \frac{-z}{2}\right), \frac{t}{8}\right), \right.$$
$$P'_{\mu,\nu}\left(\phi\left(\frac{-x}{2}, \frac{-x}{2}, \frac{-z}{2}, \frac{-z}{2}\right), \frac{t}{8}\right), P'_{\mu,\nu}\left(\phi\left(\frac{3x}{2}, \frac{3x}{2}, \frac{3z}{2}, \frac{3z}{2}\right), \frac{t}{8}\right),$$
$$P'_{\mu,\nu}\left(\phi\left(\frac{x}{2}, \frac{-x}{2}, \frac{z}{2}, \frac{-z}{2}\right), \frac{t}{8}\right), P'_{\mu,\nu}\left(\phi\left(\frac{-x}{2}, \frac{x}{2}, \frac{-z}{2}, \frac{z}{2}\right), \frac{t}{8}\right),$$
$$\left. P'_{\mu,\nu}\left(\phi\left(\frac{x}{2}, \frac{x}{2}, \frac{z}{2}, \frac{z}{2}\right), \frac{t}{8}\right), P'_{\mu,\nu}\left(\phi\left(\frac{-x}{2}, \frac{-x}{2}, \frac{-z}{2}, \frac{-z}{2}\right), \frac{t}{8}\right) \right\} \quad (3.5)$$

(using the notation of Definition 2.6)

Proof. Putting $x = \frac{x}{2}$, $y = \frac{y}{2}$, $z = \frac{z}{2}$, $w = \frac{w}{2}$ in (3.3) we get

$$P_{\mu,\nu}\left(f\left(\frac{x+y}{2}, \frac{z+w}{2}\right) - g\left(\frac{x}{2}, \frac{z}{2}\right) - h\left(\frac{y}{2}, \frac{w}{2}\right), t\right) \geq_{L^*} P'_{\mu,\nu}\left(\phi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, \frac{w}{2}\right), t\right) \tag{3.6}$$

Also putting $y = x$, $w = z$ and in (3.6) we have

$$P_{\mu,\nu}\left(f(x, z) - g\left(\frac{x}{2}, \frac{z}{2}\right) - h\left(\frac{x}{2}, \frac{z}{2}\right), t\right) \geq_{L^*} P'_{\mu,\nu}\left(\phi\left(\frac{x}{2}, \frac{x}{2}, \frac{z}{2}, \frac{z}{2}\right), t\right) \tag{3.7}$$

Replacing x by y and z by w respectively in (3.7) we have

$$P_{\mu,\nu}\left(f(y, w) - g\left(\frac{y}{2}, \frac{w}{2}\right) - h\left(\frac{y}{2}, \frac{w}{2}\right), t\right) \geq_{L^*} P'_{\mu,\nu}\left(\phi\left(\frac{y}{2}, \frac{w}{2}, \frac{y}{2}, \frac{w}{2}\right), t\right) \tag{3.8}$$

Hence using (3.6), (3.7) and (3.8) we obtain

$$\begin{aligned}
&P_{\mu,\nu}\left(2f\left(\frac{x+y}{2}, \frac{z+w}{2}\right) - f(x, z) - f(y, z), 4t\right) \\
&= P_{\mu,\nu}\left\{\left(f\left(\frac{x+y}{2}, \frac{z+w}{2}\right) - g\left(\frac{x}{2}, \frac{z}{2}\right) - h\left(\frac{y}{2}, \frac{w}{2}\right)\right) + \right. \\
&\quad \left(f\left(\frac{y+x}{2}, \frac{w+z}{2}\right) - g\left(\frac{y}{2}, \frac{w}{2}\right) - h\left(\frac{x}{2}, \frac{z}{2}\right)\right) - \\
&\quad \left(f(x, z) - g\left(\frac{x}{2}, \frac{z}{2}\right) - h\left(\frac{x}{2}, \frac{z}{2}\right)\right) - \\
&\quad \left.\left(f(y, w) - g\left(\frac{y}{2}, \frac{w}{2}\right) - h\left(\frac{y}{2}, \frac{w}{2}\right), 4t\right)\right\} \\
&\geq_{L^*} \Gamma^3\left\{P'_{\mu,\nu}\left(\phi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, \frac{w}{2}\right), t\right), P'_{\mu,\nu}\left(\phi\left(\frac{y}{2}, \frac{x}{2}, \frac{w}{2}, \frac{z}{2}\right), t\right), \right. \\
&\left. P'_{\mu,\nu}\left(\phi\left(\frac{x}{2}, \frac{x}{2}, \frac{z}{2}, \frac{z}{2}\right), t\right), P'_{\mu,\nu}\left(\phi\left(\frac{y}{2}, \frac{y}{2}, \frac{w}{2}, \frac{w}{2}\right), t\right)\right\}
\end{aligned}$$

That is,

$$\begin{aligned}
&P_{\mu,\nu}\left(2f\left(\frac{x+y}{2}, \frac{z+w}{2}\right) - f(x, z) - f(y, w), t\right) \\
&\geq_{L^*} \Gamma^3\left\{P'_{\mu,\nu}\left(\phi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, \frac{w}{2}\right), \frac{t}{4}\right), P'_{\mu,\nu}\left(\phi\left(\frac{y}{2}, \frac{x}{2}, \frac{w}{2}, \frac{z}{2}\right), \frac{t}{4}\right), \right.
\end{aligned}$$

$$P'_{\mu,\nu}\left(\phi\left(\frac{x}{2},\frac{x}{2},\frac{z}{2},\frac{z}{2}\right),\frac{t}{4}\right),P'_{\mu,\nu}\left(\phi\left(\frac{y}{2},\frac{y}{2},\frac{w}{2},\frac{w}{2}\right),\frac{t}{4}\right)\Bigg\} \quad (3.9)$$

Now putting $y=-x$ and $w=-z$ in the equation (3.9), we get

$$P_{\mu,\nu}(2f(0,0)-f(x,z)-f(-x,-z),t)$$
$$\geq_{L^*}\Gamma^3\left\{P'_{\mu,\nu}\left(\phi\left(\frac{x}{2},\frac{-x}{2},\frac{z}{2},\frac{-z}{2}\right),\frac{t}{4}\right),P'_{\mu,\nu}\left(\phi\left(\frac{-x}{2},\frac{x}{2},\frac{-z}{2},\frac{z}{2}\right),\frac{t}{4}\right),\right.$$
$$\left.P'_{\mu,\nu}\left(\phi\left(\frac{x}{2},\frac{x}{2},\frac{z}{2},\frac{z}{2}\right),\frac{t}{4}\right),P'_{\mu,\nu}\left(\phi\left(\frac{-x}{2},\frac{-x}{2},\frac{-z}{2},\frac{-z}{2}\right),\frac{t}{4}\right)\right\} \quad (3.10)$$

For our purpose let us define $F(x,z)=f(x,z)-f(0,0)$ such that it is satisfies (3.10). Clearly, $F(0,0)=0$ for all $x\in X$. So (3.10) becomes

$$P_{\mu,\nu}(-F(x,z)-F(-x,-z),t)$$
$$\geq_{L^*}\Gamma^3\left\{P'_{\mu,\nu}\left(\phi\left(\frac{x}{2},\frac{-x}{2},\frac{z}{2},\frac{-z}{2}\right),\frac{t}{4}\right),P'_{\mu,\nu}\left(\phi\left(\frac{-x}{2},\frac{x}{2},\frac{-z}{2},\frac{z}{2}\right),\frac{t}{4}\right),\right.$$
$$\left.P'_{\mu,\nu}\left(\phi\left(\frac{x}{2},\frac{x}{2},\frac{z}{2},\frac{z}{2}\right),\frac{t}{4}\right),P'_{\mu,\nu}\left(\phi\left(\frac{-x}{2},\frac{-x}{2},\frac{-z}{2},\frac{-z}{2}\right),\frac{t}{4}\right)\right\} \quad (3.11)$$

Also replacing $x=-x$, $y=3x$ and $z=-z$, $w=3z$ respectively in (3.9), we have

$$P_{\mu,\nu}(2f(x,z)-2f(0,0)-(-x,-z)+f(0,0)-F(3x,3z)+f(0,0),t)$$
$$=P_{\mu,\nu}(2F(x,z)-F(-x,-z)-F(3x,3z),t)$$
$$\geq_{L^*}\Gamma^3\left\{P'_{\mu,\nu}\left(\phi\left(\frac{-x}{2},\frac{3x}{2},\frac{-z}{2},\frac{3z}{2}\right),\frac{t}{4}\right),P'_{\mu,\nu}\left(\phi\left(\frac{3x}{2},\frac{-x}{2},\frac{3z}{2},\frac{-z}{2}\right),\frac{t}{4}\right),\right.$$
$$\left.P'_{\mu,\nu}\left(\phi\left(\frac{-x}{2},\frac{-x}{2},\frac{-z}{2},\frac{-z}{2}\right),\frac{t}{4}\right),P'_{\mu,\nu}\left(\phi\left(\frac{3x}{2},\frac{3x}{2},\frac{3z}{2},\frac{3z}{2}\right),\frac{t}{4}\right)\right\} \quad (3.12)$$

Hence we have

$$P_{\mu,\nu}(3F(x,z)-F(3x,3z),2t)$$

$$\geq_{L^*}\Gamma^7\left\{P'_{\mu,\nu}\left(\phi\left(\frac{-x}{2},\frac{3x}{2},\frac{-z}{2},\frac{3z}{2}\right),\frac{t}{4}\right),P'_{\mu,\nu}\left(\phi\left(\frac{3x}{2},\frac{-x}{2},\frac{3z}{2},\frac{-z}{2}\right),\frac{t}{4}\right),\right.$$

$$P'_{\mu,\nu}\left(\phi\left(\frac{-x}{2},\frac{-x}{2},\frac{-z}{2},\frac{-z}{2}\right),\frac{t}{4}\right),P'_{\mu,\nu}\left(\phi\left(\frac{3x}{2},\frac{3x}{2},\frac{3z}{2},\frac{3z}{2}\right),\frac{t}{4}\right),$$

$$P'_{\mu,\nu}\left(\phi\left(\frac{x}{2},\frac{-x}{2},\frac{z}{2},\frac{-z}{2}\right),\frac{t}{4}\right), P'_{\mu,\nu}\left(\phi\left(\frac{-x}{2},\frac{x}{2},\frac{-z}{2},\frac{z}{2}\right),\frac{t}{4}\right),$$

$$P'_{\mu,\nu}\left(\phi\left(\frac{x}{2},\frac{x}{2},\frac{z}{2},\frac{z}{2}\right),\frac{t}{4}\right), P'_{\mu,\nu}\left(\phi\left(\frac{-x}{2},\frac{-x}{2},\frac{-z}{2},\frac{-z}{2}\right),\frac{t}{4}\right)\Bigg\}$$

$$= M_1((x,z),2t) \tag{3.13}$$

for all $x, z \in X$, $t > 0$.

$$P_{\mu,\nu}(F(3x,3z) - 3F(x,z),t) \geq_{L^*} M_1((x,z),t) \tag{3.14}$$

Now consider the set
$E := \{G : X \times X \to Y / G(x,z) = g(x,z) - g(0,0),\ g : X \times X \to Y\}$
for all $x, z \in X$ and introduce a complete generalized metric [5] on E

$$d(G,H) = \inf\{k \in R^+ : P_{\mu,\nu}(G(x,z) - H(x,z), kt) \geq_{L^*} M_1((x,z),t)\}$$

for all $x, z \in X$, $t > 0$ and $G, H \in E$.

Consider the mapping $J : E \to E$ such that
$JF(x,z) = \frac{F(3x,3z)}{3} = \frac{f(3x,3z)-f(0,0)}{3}$ for all $F \in E$ and $x, z \in X$.
We now prove that J is a strictly contracting mapping on E with the Lipschitz constant $\frac{\alpha}{3}$.
Let $G, H \in E$ and $\epsilon > 0$. Then there exists $k' \in R^+$ satisfying
$P_{\mu,\nu}(G(x,z) - H(x,z), k't) \geq_{L^*} M_1((x,z),t)$
such that $d(G,H) \leq k' < d(G,H) + \epsilon$
Then

$$\inf\{k \in R^+ : P_{\mu,\nu}(G(x,z) - H(x,z), kt) \geq_{L^*} M_1((x,z),t)\}$$

$$\leq k' < d(G,H) + \epsilon$$

that is,

$$\inf\left\{k \in R^+ : P_{\mu,\nu}\left(\frac{G(3x,3z)}{3} - \frac{H(3x,3z)}{3}, \frac{kt}{3}\right) \geq_{L^*} M_1((3x,3z),t)\right\}$$

$$\leq d(G,H) + \epsilon$$

that is,

$$\inf\left\{k \in R^+ : P_{\mu,\nu}\left(JG(x,z) - JH(x,z), \frac{kt}{3}\right) \geq_{L^*} M_1(3x,3z),t)\right\}$$

$$\leq d(G, H) + \epsilon$$

that is,

$$\inf \left\{ k \in R^{+} : P_{\mu,\nu}(JG(x, z) - JH(x, z), \frac{k\,\alpha\,t}{3}) \geq_{L^*} M_1((x, z), t) \right\}$$
$$\leq d(g, h) + \epsilon$$

as $M_1((3^n x, 3^n z), t) = M_1((x, z), \frac{t}{\alpha^n})$
or, $d\left\{ \frac{3}{\alpha}(JG, JH)\right\} < d(G, H) + \epsilon$
or, $d\{(JG, JH)\} < \frac{\alpha}{3}\{d(G, H) + \epsilon\}$
Taking $\epsilon \to 0$ we get $d\{(JG, JH)\} < \frac{\alpha}{3}\{d(G, H)\}$
Therefore J is strictly contractive mapping with Lipschitz constant $\frac{\alpha}{3}$.
Also from (3.14) $d(F, JF) \leq \frac{1}{3}$
and $d(JF, J^2 F) \leq \frac{\alpha}{3} d(F, JF) < \infty$
Again replacing x by $3^n x$ in (3.14) we get
$P_{\mu,\nu}(F(3^{n+1}x, 3^{n+1}z) - 3F(3^n x, 3^n z), t) \geq_{L^*} M_1((3^n x, 3^n z), t)$
or, $P_{\mu,\nu}(\frac{F(3^{n+1}x\, 3^{n+1}z)}{3^{n+1}} - \frac{F(3^n x, 3^n z)}{3^n}, \frac{t}{3^{n+1}}) \geq_{L^*} M_1((3^n x, 3^n z), t)$

$$\geq_{L^*} M_1((x, z), \frac{t}{\alpha^n})$$

or, $P_{\mu,\nu}\left(J^{n+1}F(x, z) - J^n F(x, z), t\frac{(\frac{\alpha}{3})^n}{3}\right) \geq_{L^*} M_1((x, z), t)$
Hence $d(J^{n+1}F, J^n F) \leq \frac{1}{3}(\frac{\alpha}{3})^n < \infty$ as Lipschitz constant $\frac{\alpha}{3} < 1$ for $n \geq n_0 = 1$.

Therefore by Theorem 2.14 there exists a mapping $A : X \times X \to Y$ such that the following holds
1. A is a fixed point of J, that is, $A(3x, 3z) = 3A(x, z)$ for all $x, z \in X$.
The mapping A is a unique fixed point of J in the set
$E_1 = \{G \in E : d(J^{n_0} F, G) = d(JF, G) < \infty\}$
Therefore $d(JF, A) < \infty$.
Also from (3.14) $d(JF, F) \leq \frac{1}{3} < \infty$
Thus $F \in E_1$
Now, $d(F, A) \leq d(F, JF) + d(JF, A) \leq \infty$.
Thus there exists $k \in (0, \infty)$ satisfying

$$P_{\mu,\nu}(F(x, z) - A(x, z), kt) \geq_{L^*} M_1((x, z), t)$$

for all $x, z \in X, t > 0$;
that is,

$$P_{\mu,\nu}(f(x, z) - f(0, 0) - A(x, z), kt) \geq_{L^*} M_1((x, x), t)$$

Also

$$P_{\mu,\nu}(F(3^n x, 3^n z) - A(3^n x, 3^n z), kt) \geq_{L^*} M_1((3^n x, 3^n z), t)$$

or,

$$P_{\mu,\nu}\left(\frac{F(3^n x, 3^n z)}{3^n} - \frac{A(3^n x, 3^n z)}{3^n}, \frac{kt}{3^n}\right) \geq_{L^*} M_1(\alpha^n (x, z), t)$$

or,

$$P_{\mu,\nu}\left(J^n F(x, z) - \frac{A(3^n x, 3^n z)}{3^n}, \frac{k\alpha^n t}{3^n}\right) \geq_{L^*} M_1((x, z), t)$$

2. $d(J^n F, A)$
$= \inf\{k \in R^+ : P_{\mu,\nu}(J^n F(x, z) - A(x, z), (\frac{\alpha}{3})^n kt) \geq_{L^*} M_1((x, z), t\},$
since, $A(3^n x, 3^n x) = 3^n A(x, x)$
Therefore $d(J^n F, A) \leq (\frac{\alpha}{3})^n k \to 0$ as $n \to \infty$. This implies the equality

$$A(x, z) = \lim_{n\to\infty} J^n F(x, z) = \lim_{n\to\infty} \frac{F(3^n x, 3^n z)}{3^n}$$

$$= \lim_{n\to\infty} \frac{f(3^n x, 3^n z) - f(0, 0)}{3^n} \tag{3.15}$$

for all $x, z \in X$.

3. $d(F, A) \leqslant \frac{1}{1-L} d(F, JF)$ with $F \in E_1$ which implies the inequality

$$d(F, A) \leq \frac{1}{1-\frac{\alpha}{3}} \times \frac{1}{3} = \frac{1}{3-\alpha}$$

then it follows that

$$P_{\mu,\nu}(A(x, z) - F(x, z), \frac{1}{3-\alpha} t) \geq_{L^*} M_1((x, z, t)$$

It implies that

$$P_{\mu,\nu}(A(x, z) - F(x, z), t) \geq_{L^*} M_1((x, z), (3-\alpha)t)$$

That is,

$$P_{\mu,\nu}(A(x, z) - f(x, z) - f(0, 0), t) \geq_{L^*} M_1((x, z), (3-\alpha)t) \tag{3.16}$$

for all $x, z \in X; t > 0$.
Also from the definition of A we have

$$A(3x, 3z) = 3A(x, z) \textit{ and } A(0, 0) = 0 \tag{3.17}$$

Now

$$P_{\mu,\nu}(2A(2x, 2z) - 4A(x, z), t)$$

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$$= P_{\mu,\nu}\Bigg(\Bigg(2\,A(2x,\,2z) - \frac{2F(3^n\,2x,\,3^n\,2z)}{3^n} - A(3x,\,3z) + \frac{F(3^{n+1}\,x,\,3^{n+1}\,z)}{3^n} - A(x,\,z) + \frac{F(3^n\,x,\,3^n\,z)}{3^n}\Bigg)$$

$$+ \left(\frac{2F(3^n\,2x,\,3^n\,2z)}{3^n} - \frac{F(3^{n+1}\,x,\,3^{n+1}\,z)}{3^n} - \frac{F(3^n\,x,\,3^n\,z)}{3^n}\right), t\Bigg)$$

$$\geq_{L^*} \Gamma^3 \left\{ P_{\mu,\nu}\left(2\,A(2x,\,2z) - \frac{2F(3^n\,2x,\,3^n\,2z)}{3^n}, \frac{t}{4}\right), \right.$$

$$P_{\mu,\nu}\left(A(3x,\,3z) - \frac{F(3^{n+1}\,x,\,3^{n+1}\,z)}{3^n}, \frac{t}{4}\right), P_{\mu,\nu}\left(A(x,\,z) - \frac{F(3^n\,x,\,3^n\,z)}{3^n}, \frac{t}{4}\right),$$

$$\left. P_{\mu,\nu}\left(\frac{2F(3^n\,2x,\,3^n\,2z)}{3^n} - \frac{F(3^{n+1}\,x,\,3^{n+1}\,z)}{3^n} - \frac{F(3^n\,x,\,3^n\,z)}{3^n}, \frac{t}{4}\right)\right)$$

$$= \Gamma^3 \left\{ P_{\mu,\nu}\left(2\,A(2x,\,2z) - \frac{2F(3^n\,2x,\,3^n\,2z)}{3^n}, \frac{t}{4}\right), \right.$$

$$P_{\mu,\nu}\left(A(3x,\,3z) - \frac{F(3^{n+1}\,x,\,3^{n+1}\,z)}{3^n}, \frac{t}{4}\right), P_{\mu,\nu}\left(A(x,\,z) - \frac{F(3^n\,x,\,3^n\,z)}{3^n}, \frac{t}{4}\right),$$

$$\left. P_{\mu,\nu}\left(2F(3^n\,2x,\,3^n\,2z) - F(3^{n+1}\,x,\,3^{n+1}\,z) - F(3^n\,x,\,3^n\,z), \frac{3^n\,t}{4}\right)\right)$$

Taking the limit as $n \to \infty$ in the above and using (3.5), (3.15) (3.17) and for the last term using (3.9) and (3.2),
we have

$$P_{\mu,\nu}\left(2\,A(2x,\,2z) - 4A(x,\,z),\,t\right) = 1_{L^*}$$

That is, $A(2x,\,2z) = 2A(x,\,z)$
Therefore,

$$P_{\mu,\nu}\left(A(x+y,\,z+w) - A(x,\,z) - A(y,\,w),\,t\right)$$

$$= P_{\mu,\nu}\left(2A\left(\frac{x+y}{2}, \frac{z+w}{2}\right) - A(x,\,z) - A(y,\,w),\,t\right)$$

and similarly as above it can be proved that

$$P_{\mu,\nu}\left(A(x+y,\,z+w) - A(x,\,z) - A(y,\,w),\,t\right) = 1_{L^*}$$

for all $x,\,y \in X$.
That is,
$A(x+y,\,z+w) = A(x,\,z) + A(y,\,w)$, that is A is additive.

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The uniqueness of A follows from the fact that A is the unique fixed point of J with the following property that there exists $u \in (0, \infty)$ such that

$$P_{\mu,\nu}(f(x, z) - f(0, 0) - A(x, z), ut) \geq_{L^*} M_1((x, z), t)$$

for all $x, z \in X$ and $t > 0$. This completes the proof of the theorem. □

Corollary 3.2. Let $p < 1$ be a non-negative real number and X be norm linear space with norm $\|.\|$, $(Z, P'_{\mu,\nu}, M)$ be an IFN-space, $(Y, P_{\mu,\nu}, M)$ be a complete IFS-space and $z_0 \in Z$. If $f, g, h : X \times X \to Y$ are mappings such that

$$P_{\mu,\nu}(f(x+y, z+w) - g(x, z) - h(y, w), t) \geq_{L*} P'_{\mu,\nu}(z_0(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p), t)$$

$$(x, y, z, w \in X, t > 0, z_0 \in Z)$$

then there exists a unique additive mapping $A : X \times X \to Y$ such that $P_{\mu,\nu}(f(x, z) - f(0, 0) - A(x, z), t) \geq_{L^*} P'_{\mu,\nu}\left(z_0(\|x\| + \|z\|)^p, \frac{2^p t}{16.3^p}(3 - 3^p)\right)$

for all $x \in X$ and $t > 0$, $z_0 \in Z$.

Proof : Define $\phi(x, y, z, w) = z_0(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$ and it can be proved by similar way as Theorem 3.1 by $\alpha = 3^p$, where $0 < \alpha < 3$.

Theorem 3.3. *Let X be a linear space, $(Z, P'_{\mu,\nu}, \Gamma)$ be a IFN-space, $\phi : X \times X \times X \times X \to Z$ be a function such that*

$$P'_{\mu,\nu}\left(\phi\left(\frac{x}{3}, \frac{y}{3}, \frac{z}{3}\frac{w}{3}\right), t\right) \geq_{L^*} P'_{\mu,\nu}\left(\frac{1}{\alpha}\phi(x, y, z, w), t\right) \quad (3.18)$$

for some real α with $\alpha > 3$, $(x, y, z, w \in X, t > 0)$ and

$$\lim_{n\to\infty} P'_{\mu,\nu}(\phi(3^n x, 3^n y, 3^n z, 3^n w), 3^n t) = 1_{L^*}$$

for all $x, y, z, w \in X$ and $t > 0$. Let $(Y, P_{\mu,\nu}, \Gamma)$ be a complete IFN-space. If $f, g, h : X \times X \to Y$ are mappings such that

$$P_{\mu,\nu}(f(x+y, z+w) - g(x, z) - h(y, w), t) \geq_{L^*} P'_{\mu,\nu}(\phi(x, y, z, w), t) \quad (3.19)$$

$(x, y, z, w \in X, , t > 0)$.
Then there exists a unique additive mapping $A : X \times X \to Y$ define by $A(x, z) := \lim_{n\to\infty}\left(\frac{f(3^n x, 3^n z) - f(0,0)}{3^n}\right)$ for all $x, z \in X$ satisfying

$$P_{\mu,\nu}(f(x, z) - f(0, 0) - A(x, z), t) \geq_{L^*} M_1((x, z), t(\alpha - 3)) \quad (3.20)$$

where $M_1((x, z), t)$ is given in the Theorem 3.1

Example 3.4. Let $(X, \|.\|)$ be any real Banach space, M a continuous t-norm defined in Example 2.8. Then $(X, P_{\mu,\nu}, M)$ is a complete IFN-space in which $P_{\mu,\nu}(x, t) = \left(\frac{t}{t+||x||}, \frac{||x||}{t+||x||})\right)$.

we define $f, g, h : X \times X \to X$, by
$f(x, y) = ax + by + \|x\|x_0 + \|y\|x_0,$
$g(x, y) = cx + dy + \|x\|x_0 + \|y\|x_0,$
$h(x, y) = ex + fy + \|x\|x_0 + \|y\|x_0,$
where x_0 is unit vectors in X and a, b, c, d, e, f are positive real numbers.
Then $\|f(x+y, z+w) - g(x, z) - h(y, w)\|$
$\leq |a-c|\|x\| + |a-e|\|y\| + |b-d|\|z\| + |b-f|\|w\|$
for all $x, y, z, w \in X$.
Thus $P_{\mu,\nu}(f(x+y, z+w) - g(x, z) - h(y, w), t)$
$\geq_{L^*} P_{\mu,\nu}((|a-c|\|x\| + |a-e|\|y\| + |b-d|\|z\| + |b-f|\|w\|)x_0, t)$
for all $x, y, z, w \in X$ and $t > 0$.
Let $\phi(x, y, z, w) = (|a-c|\|x\| + |a-e|\|y\| + |b-d|\|z\| + |b-f|\|w\|)x_0$
for all $x, y, z, w \in X$.
Also $P_{\mu,\nu}(\phi(3x, 3y, 3z, 3w), t) \geq P_{\mu,\nu}(3\phi(x, y, z, w), t)$ for all $x, y, z, w \in X$ and $t > 0$. Hence all the conditions of Theorem 3.1 hold. Therefore f can be approximated by as additive mapping. Actually there exists a unique additive mapping $A : X \times X \to X$ such that

$P_{\mu,\nu}(f(x, z) - f(0, 0) - A(x, z), t)$

$\geq_{L^*} P_{\mu,\nu}\left(x_0, \dfrac{t}{4\alpha}\right)$

Where

$\alpha = \min\{((4a+3e+c)||x|| + (4b+3f+d)||z||), ((4a+3c+e)||x|| + (4b+f+3d)||z||),$
$((2a+e+c)||x|| + (2b+f+d)||z||), 3((2a+c+e)||x|| + (2b+d+f)||z||)\}$
for all $x, y, z, w \in X$ and $t > 0$.

4 Conclusion

There are several studies in functional analysis which have been extended to the fuzzy linear spaces and their extensions. In this paper we have conducted such a study in intuitionistic normed linear spaces. The Hyers-Ulam-Rassias stability of several other functional equations may also be taken up in these spaces. This can be a future program. Also the fixed point approach that we have used in this paper can possibly be applied elsewhere.

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References

[1] K. T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, 20 (1986), 87−96.

[2] J. H. Bae and W. G. Park, *Stability of a quadratic functional equation in intuitionistic fuzzy normed spaces*, Commun. Korean Math. Soc., 26 (2011), 237−251.

[3] A. Branciari, *A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces* , Publ. Math. Debrecen., 57 (2000), 31−37.

[4] J. J. Buckley, *Generalized and extended fuzzy sets with applications* , Fuzzy Sets and Systems, 25 (1988), 159−174.

[5] L. Cadariu, V. Radu, *Fixed points and stability for functional equations in probabilistic metric and random normed spaces*, Fixed point theory and applications, Article ID 589143, 18 pages, 2009 (2009).

[6] J. Chmielinski and J. Tabor, On approximate solutons of the Pexider equation, Aequ. Math., 46 (1993), 143−163.

[7] G. Deschrijiver, E. E. Kerre, *On the relationship between some extensions of fuzzy set theory*, Fuzzy Sets and Systems, 23 (2003), 227−235.

[8] J. B. Diaz and B. Margolisi, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soci, 74 (1968), 305−309.

[9] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci., U.S.A., 27 (1941), 222−224.

[10] N. C. Kayal, P. Mondal, T. K. Samanta *On stability of a Pexiderized functional equation in intuitionistic fuzzy banach spaces*, Application and Applied Mathematics, 10 (2015), 783−794.

[11] N. C. Kayal, T. K. Samanta, P. Saha, B. S. Choudhury, *A Hyers-Ulam-Rassias stability result for functional equations in intuitionistic fuzzy banach spaces*, Iranian Journal Fuzzy Systems, 13 (2016), 87−96.

[12] D. Mihet, *The fixed point method for fuzzy stability of the Jensen functional equation*, Fuzzy sets and systems, 160 (2009), 1663−1667.

[13] P. Mondal, N. C. Kayal, T. K. Samanta, *The stability of pexider type functional equation in intuitionistic fuzzy banach spaces via fixed point technique*, J. Hyperstructures, 4 (2015), 37−49.

[14] P. Mondal, N. C. Kayal, T. K. Samanta, *Stability of a quadratic functional equation in intuitionistic fuzzy banach spaces*, J. New Results. Sci., 10 (2016), 52−59.

[15] J. H. Park, *Intuitionistic fuzzy metric spaces*, Chaos, Solitons and Fractals, 22 (2004), 1039−1046.

[16] Th. M. Rassias, *On the stability of the functional equations in Banach spaces*, J. Math. Anal. Appl., 251 (2000), 264−284.

[17] R. Saadati, J. H. Park, *On Intuitionistic fuzzy topological spaces*, Chaos, Solitons and Fractals, 27 (2006), 331−344.

[18] S. Shakeri, *Intutionistic fuzzy stability of Jenson type mapping*, J. Nonlinear Sci. Appl., 2 (2009), 105−112.

[19] V. Torra, Y. Narukawa, *On hesitant fuzzy sets and decision*, In: The 18th IEEE international conference on fuzzy systems, Jeju island, Korea, 544 (2009), 1378−1382.

[20] S. M. Ulam, *Problems in Modern Mathematics*, Chapter vi, Science Editions, Wiley, New York, 1964.

[21] Z. Wang, T. M. Rassias, R. Saadati, *Intuitionistic fuzzy stability of Jensen-type quadratic functional equations*, Filomat, 28 (2014), 663−676.

[22] L. A. Zadeh, *Fuzzy sets*, Information and Control, 8 (1965), 338−353.