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Relations for the moments of dual generalised order statistics from exponentiated Pareto type I distribution

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Abstract

The exponentiated Pareto type I distribution is used as a model for the distribution of city populations within a given area. In this paper, we have considered the dual generalized order statistics from this distribution. We obtained exact explicit expressions as well as recurrence relations for single and product moments of dual generalized order statistics from generalized Pareto distribution. Then, we use these results, we have tabulated the first four moments, skewness and kurtosis of order statistics from samples of sizes up to 10 for various values of the parameters.

Keywords: Sample; dual generalized order statistics; record values; single and product moments; recurrence relations; exponentiated Pareto type I distribution.

1 Introduction

The Pareto distribution was first proposed as a model for the distribution of incomes. It is also used as a model for the distribution of city populations within a given area. The Pareto distribution is a skewed, heavy-tailed distribution that is sometimes used to model the distribution of incomes and other financial variables. Pareto distributions are very versatile and a variety of uncertainties can be usefully modeled by them. For instance, they arise as tractable life time models in actuarial science, economics, finance, life testing, survival analysis, and telecommunications. Many applications of the Pareto distribution in economics, biology and physics can be found throughout the literature. Burroughs and Tebbens (2001) discussed applications of the Pareto distribution in modeling earthquakes, forest fire areas and oil and gas field sizes, and Schroeder, et al. (2010) presented an application of the Pareto distribution in modeling disk drive sector errors. To add flexibility to the Pareto distribution, various generalizations of the distribution have been derived, including: the generalized Pareto distribution (Pickands, 1975), the beta-Pareto distribution (Akinsete, et al., 2008) and the beta generalized Pareto distribution (Mahmoudi, 2011). One of the important families of distributions in lifetime tests is the exponentiated Pareto distribution. The exponentiated Pareto distribution has been introduced by Gupta et al. (1998). Khan and Kumar (2010) discussed the exponentiated Pareto distribution as an important model of life time models and derived the explicit expressions and some recurrence relations for single and product moments of lower generalized order statistics. The Pareto distribution has probability density function

$$f(x;\alpha,\beta) \;=\; \frac{\alpha\beta^{\alpha}}{(x+\beta)^{\alpha+1}}, \quad x>0, \quad \alpha,\beta>0. \tag{1}$$

and the corresponding cumulative distribution function is

$$F(x;\alpha,\beta) \;=\; 1-\left(\frac{\beta}{x+\beta}\right)^{\alpha}, \quad x>0, \quad \alpha,\beta>0. \tag{2}$$

where β is a scale parameter and α is the shape parameter. Consider the transformation $Y = X+\beta$ to get another form of the Pareto distribution

$$f(x;\alpha,\beta) \;=\; \frac{\alpha\beta^{\alpha}}{x^{\alpha+1}}, \quad \beta\le x<\infty, \quad \alpha,\beta>0. \tag{3}$$

Kumar (2013, 2014,) have established the recurrence relations for moment of lower generalized order statistics and generalized order statistics from exponentiated Lomax and extended type II generalized Logistic distribution distribution respectively. Also Kumar and Kulshrestha (2013) obtained relations for moments of record values from generalized Pareto distribution and Kumar (2015) have established the relations for moments of record values from generalized rayleigh distribution and Khan et al. (2012) obtained the moment generating function of lower generalized order statistics from generalized exponential distribution.
Order statistics and functions of these statistics play an important role in a wide range of theoretical and practical problems such as characterization of probability distributions and goodness-of-fit tests, entropy estimation, analysis of censored samples, reliability analysis, quality control and strength of materials; see Arnold et al. (1992) and David and Nagaraja (2003) and the references therein for more details. The practicability of moments of order statistics can be seen in many areas such as quality control testing, reliability, etc. For instance, when the reliability of an item or product is high, the duration of the failed items will be high which in turn will make the product too expensive, both in terms of time and money. This fact prevents a practitioner from knowing enough about the product in a relatively short time. Therefore, a practitioner needs to predict the failure of future items based on the times of a few early failures. These predictions are often based on moments of order statistics.

In this article, we define dual generalized order statistics. It will be shown that order statistics, record values and progressively Type II censored order statistics are special cases of dual generalized order statistics. First, we derive the explicit expressions for single moments and product moments order statistics and record values. Further, the characterization of this distribution by using recurrence relations and conditional moments of *dgos*. We also provide tabulations of first four moments, skewness and kurtosis of order statistics and record values for samples of different sizes of the shape and scale parameters.

The remaining of the article is organized as follows. In Section 2, we introduce the exponentiated Pareto type I distribution. In section 4, we derive relations for single and product moments of gos from exponentiated Pareto type I distribution. The obtained relations were used to compute first four moments and variances of order statistics and record values. We obtained the characterization of this distribution by using conditional moments of dual generalized order statistics in Section 5. In Section 6, numreical computations are shown for various values of α. Section 7 ends with concluding remarks.

2 Exponentiated Pareto Type I Distribution

A random variable X is said to have exponentiated Pareto type I $EPTI$ distribution if its *pddf* is given by

$$f(x) = \frac{\lambda\alpha\beta^{\alpha}}{x^{\alpha+1}}[1-(\beta/x)^{\alpha}]^{\lambda-1}, \quad x \geq \beta, \quad \alpha, \beta, \lambda > 0. \tag{4}$$

and its corresponding is

$$F(x) = [1-(\beta/x)^{\alpha}]^{\lambda}, \quad x \geq \beta, \quad \alpha, \beta, \lambda > 0. \tag{5}$$

Therefore, in view of (4) and (5), we have

$$\alpha\lambda F(x) = \left(\frac{x^{\alpha+1}}{\beta^{\alpha}} - x\right) f(x). \tag{6}$$

For $\lambda = 1$, (5) reduces to Pareto distribution. Plots of the density for selected parameter values are shown in Figure 1. It can be observed that the $EPTI$ distribution is capable to modeling various types of data as it can take various shapes (Leptokurtic, platykurtic with thin tails) for various choices of the parameters.

2.1 Hazard rate function

The basic tools for studying the ageing and reliability characteristics of the system are the hazard rate (HR) and the mean residual lifetime (MRL). The HR and the MRL deal with the residual lifetime of the system. The HR gives the rate of failure of the system immediately after time$x$, and the MRL measures the expected value of the remaining lifetime of the system, provided that it has survived up to time x. Thus the hazard rate function of $EPTI$ is given by

$$\begin{aligned} h(x) &= \frac{f(x)}{1-F(x)} \\ &= \frac{\lambda\alpha\beta^{\alpha}x^{-(\alpha+1)}[1-(\beta/x)^{\alpha}]^{\lambda-1}}{1-[1-(\beta/x)^{\alpha}]^{\lambda}}, \quad x \geq \beta, \quad \alpha, \beta, \lambda > 0. \end{aligned} \tag{7}$$

Figure 2 indicates that the hazard rate function of the $EPTI$ distribution is uni-modal or upsidedown bathtub shaped depending on the parameter values.

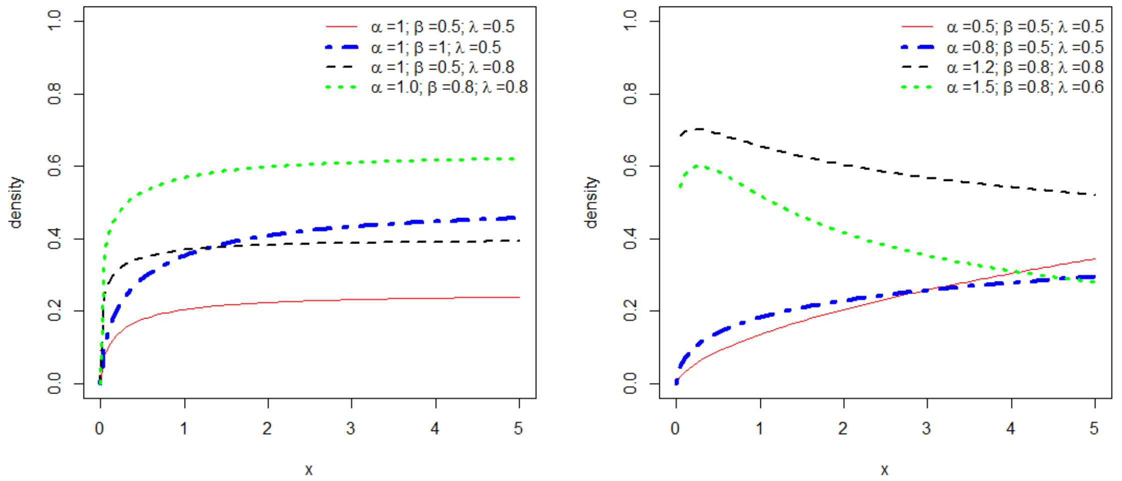

Figure 1: Probability density function of $EPTI$ distribution for different values of α, β and λ.

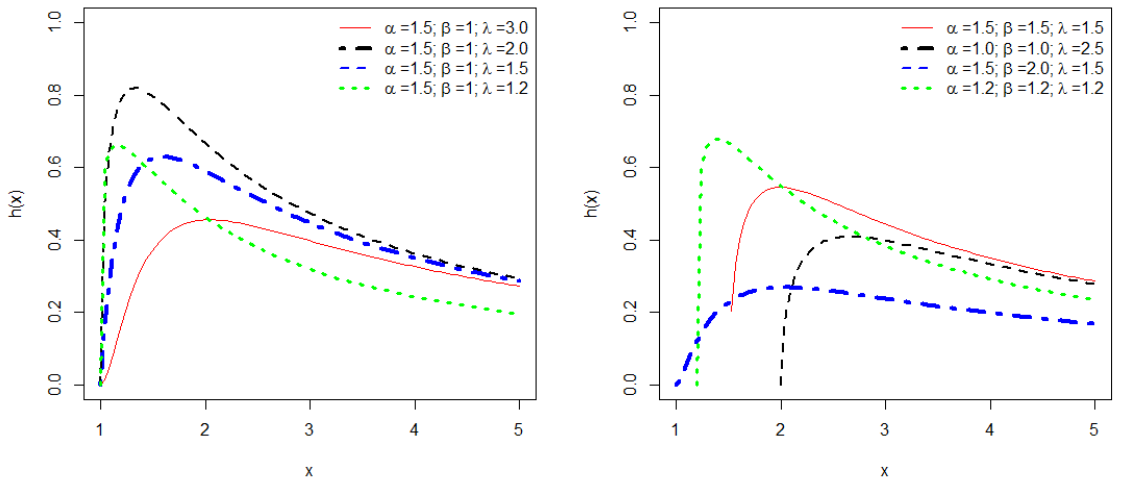

Figure 2: Hazard rate function of $EPTI$ distribution for different values of α, β and λ.

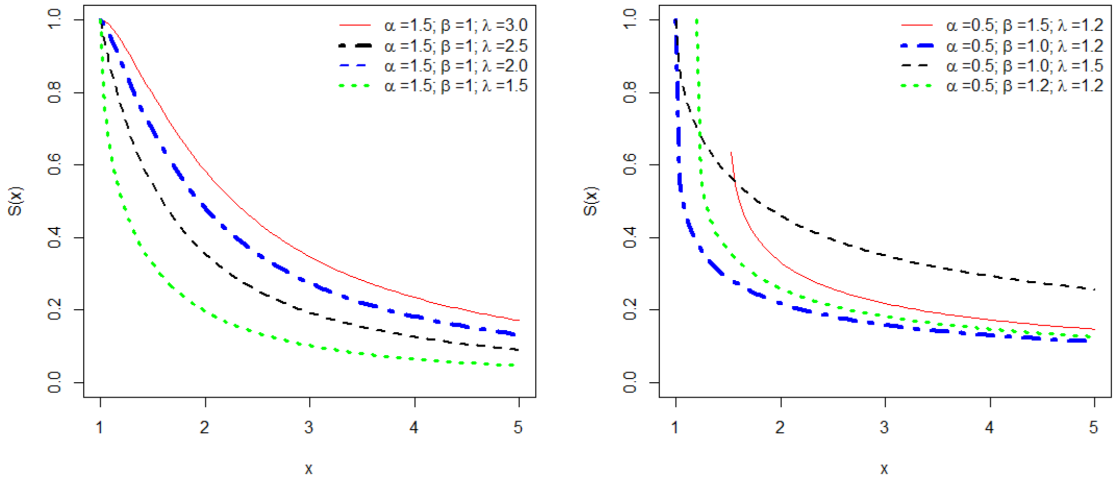

Figure 3: Survival function of $EPTI$ distribution for different values of α, β and λ.

3 Generalised order statistics and preliminaries

The concept of generalized order statistics (GOS) was introduced by Kamps (1995) as a general framework for models of ordered random variables ($rv's$). Moreover, many other models of ordered $rv's$, such as, ordinary order statistics (oos), order statistics with non-integral sample size ($nonI$), progressively type-II censored order statistics (PCO), upper record values, upper Pfeifer records and sequential order statistics (sos) are seen to be particular cases of gos. These models can be effectively applied, e.g., in reliability theory. However, decreasingly ordered $rv's$ cannot be integrated into this framework. Consequently, this model is inappropriate to study, e.g. reversed ordered order statistic and lower record values models. Using the concept of GOS Pawlas and Szynal (2001) introduced the concept of dual generalized order statistics (DGOS) to enable a common approach to descending order statistics, which was further studied by Burkschat *et al.* (2003) with the name dual generalized order statistics. The DGOS models enable us to study decreasingly ordered random variables like reversed order statistics, lower record values and lower Pfeifer records, through a common approach below:

Suppose $X'(1,n,m,k),\ldots,X'(n,n,m,k)$, $(k \geq 1,\ m$ is a real number), are n $lgos$ from an absolutely continuous cumulative distribution function (cdf) $F(x)$ with probability density function (pdf) $f(x)$. Their joint pdf of the form

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[F(x_i)]^m f(x_i)\right)[F(x_n)]^{k-1}f(x_n), \tag{8}$$

for $F^{-1}(1) > x_1 \geq x_2 \geq \ldots \geq x_n > F^{-1}(0)$, where $\gamma_j = k + (n-j)(m+1) > 0$ for all j, $1 \leq j \leq n$, k is a positive integer and $m \geq -1$. If $m = 0$ and $k = 1$, we obtain the joint pdf of the ordinary order statistics. If $k = 1$ and $m = -1$, we obtain the joint pdf of the first n

record values of the identically and independently distributed (iid) random variables with *cdf* $F(x)$ and corresponding *pdf* $f(x)$. Other statistics contained as particular cases include *sos*, progressively type II censored order statistics.

In view of (8), the marginal *pdf* of the r^{th} *lgos*, $X'(r,n,m,k)$ $1 \le r \le n$ is

$$f_{X'(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[F(x)]^{\gamma_r-1} f(x) g_m^{r-1}(F(x)). \tag{9}$$

The joint *pdf* of $X'(r,n,m,k)$ and $X'(s,n,m,k)$, $1 \le r < s \le n$ is

$$f_{X'(r,n,m,k),X'(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}[F(x)]^m f(x) g_m^{r-1}(F(x))$$
$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s-1} f(y), \quad x > y, \tag{10}$$

where $C_{r-1} = \prod_{i=1}^{r} \gamma_i$, $h_m(x) = -lnx$, if $m = -1$, $h_m(x) = \frac{1}{m+1}x^{m+1}$, if $m \ne -1$ and $g_m(x) = h_m(x) - h_m(1)$, $x \in [0,1)$.

4 Relations for single and product Moments of dual generalized order statistics

In this section, we derive explicit expressions and recurrence relations for single and product moments of dual generalized order statistics from the exponentiated Pareto type I distribution.

4.1 Relations for single moments

We shall first establish explicit expressions for single moments of jth DGOS, $E\left[X'^{(j)}(r,n,m,k)\right] = \mu^{(j)}_{r,n,m,k}$. When $m \ne -1$

$$\begin{aligned}\mu^{(j)}_{r,n,m,k} &= \frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j (F(x))^{\gamma_r-1} f(x) g_m^{r-1}(F(x))dx \\ &= \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}}\sum_{u=0}^{r-1}(-1)^u \binom{r-1}{u}\int_0^\infty x^j (F(x))^{\gamma_{r-u}-1} f(x)dx \end{aligned} \tag{11}$$

By setting $t = [F(x)]^{1/\lambda}$ in (11) and simplifying the resulting expression, we obtain

$$\begin{aligned}\mu^{(j)}_{r,n,m,k} &= \frac{\beta^j C_{r-1}}{(r-1)!(m+1)^r}\sum_{u=0}^{r-1}(-1)^u \binom{r-1}{u}\sum_{p=0}^{\infty}\frac{(j/\alpha)_p}{p!} \\ &\times \ B\left(\frac{k}{m+1} + n - r + u + \frac{p}{\lambda(m+1)}, 1\right), \end{aligned} \tag{12}$$

where

$$a_{(i)} = \begin{cases} a(a+1)...(a+i-1), & \text{if } \ i > 0, \\ 1, & \text{if } \ i = 0. \end{cases}$$

Now using the relation,

$$\sum_{a=0}^{\infty}(-1)^a B(a+k,c) = B(k,c+b).$$

we have

$$\begin{aligned}
\mu_{r,n,m,k}^{(j)} &= \frac{\beta^j C_{r-1}}{(r-1)!(m+1)^r}\sum_{p=0}^{\infty}\frac{(j/\alpha)_p}{p!}\ B\left(\frac{k}{m+1}+n-r+\frac{p}{\lambda(m+1)},r\right) \\
&= \beta^j\sum_{p=0}^{\infty}\frac{(j/\alpha)_p}{p!}\frac{1}{\prod_{a=1}^{r}\left(1+\frac{p}{\lambda\gamma_{r-a}}\right)},\quad j=1,2,... \qquad (13)
\end{aligned}$$

and when $m=-1$ that

$$\mu_{r,n,-1,k}^{(j)} = (\lambda k)^r\beta^j\sum_{p=0}^{\infty}\frac{(j/\alpha)_p}{p!(\lambda k+p)^r},\quad j=0,1,..... \qquad (14)$$

Special cases
i) Putting $m=0,\ k=1$ in (13), the explicit formula for single moments of order statistics of the $EPTI$ distribution can be obtained as

$$\mu_{n-r+1:n}^{(j)} = \beta^j\ C_{r:n}\sum_{p=0}^{\infty}\frac{(j/\alpha)_p}{p!}\frac{1}{\prod_{a=1}^{r}\left(1+\frac{p}{\lambda(n-r+a+1)}\right)}.$$

That is

$$\mu_{r:n}^{(j)} = \beta^j\ C_{r:n}\sum_{p=0}^{\infty}\frac{(j/\alpha)_p}{p!}\frac{1}{\prod_{a=1}^{r}\left(1+\frac{p}{\lambda(r+a)}\right)},$$

where

$$C_{r:n} = \frac{n!}{(r-1)!\ (n-r)!}.$$

In particular, the mean order statistics and the variance order statistics are

$$\mu_{r:n} = \beta\ C_{r:n}\sum_{p=0}^{\infty}\frac{(1/\alpha)_p}{p!}\frac{1}{\prod_{a=1}^{r}\left(1+\frac{p}{\lambda(r+a)}\right)}, \qquad (15)$$

and

$$\sigma_{r:n}^2 = \mu_{r:n}^{(2)} - \left[\mu_{r:n}^{(1)}\right]^2 \qquad (16)$$

respectively. If $\lambda=1$, then (15) and (16) reduces for the mean and the variance of order statistics of the Pareto distribution.

ii) Putting $k=1$ in (14), we deduce the explicit expression for the moments of lower record values for the $EPTI$ distribution.

$$\mu_{L(r)}^{(j)} = \lambda^r \beta^j \sum_{p=0}^{\infty} \frac{(j/\alpha)_p}{p!(\lambda+p)^r},$$

which is the result obtained by Kumar and Kumar (2017) for $r=n$.

In particular, the mean record statistics and the variance record statistics are

$$\mu_{L(r)} = \lambda^r \beta \sum_{p=0}^{\infty} \frac{(1/\alpha)_p}{p!(\lambda+p)^r}, \tag{17}$$

and

$$\sigma_{U(r)}^2 = \mu_{U(r)}^{(2)} - \left[\mu_{U(r)}^{(1)}\right]^2, \tag{18}$$

respectively. If $\lambda = 1$ then (17) and (18) reduces for the mean and the variance of record values of the Pareto distribution.

Recurrence relations for single moments of DGOS can be obtained in the following theorems.

Before arriving at the main results, we mention the following relations proved by Khan et al. (2008) which will be used in sequel. For $2 \le r \le n$, $n \ge 2$ and $k = 1, 2, ...$
i)

$$\mu_{r,n,m,k}^{(j)} - \mu_{r-1,n,m,k}^{(j)} = -\frac{jC_{r-1}}{(r-1)!(\gamma_r} \int_{\alpha}^{\beta} x^{j-1} [F(x)]^{\gamma_r} g_m^{r-1}(F(x))dx. \tag{19}$$

ii)

$$\mu_{r-1,n,m,k}^{(j)} - \mu_{r-1,n-1,m,k}^{(j)} = \frac{(m+1)jC_{r-2}}{(r-2)!(\gamma_1} \int_{\alpha}^{\beta} x^{j-1} [F(x)]^{\gamma_r} g_m^{r-1}(F(x))dx. \tag{20}$$

iii)

$$\mu_{r,n,m,k}^{(j)} - \mu_{r-1,n-1,m,k}^{(j)} = -\frac{jC_{r-1}}{(r-1)!(\gamma_1} \int_{\alpha}^{\beta} x^{j-1} [F(x)]^{\gamma_r} g_m^{r-1}(F(x))dx. \tag{21}$$

Theorem 1. *For the distribution given in (4) and* $2 \le r \le n$, $n \ge 2$ *and* $k = 1, 2, ...$

$$\left(1 - \frac{j}{\lambda\alpha\gamma_r}\right) \mu_{r,n,m,k}^{(j)} = \mu_{r-1,n,m,k}^{(j)} - \frac{j}{\alpha\lambda\beta^\alpha\gamma_r} \mu_{r,n,m,k}^{(j+\alpha)}. \tag{22}$$

Throughout, we follow the conventions that $\mu_{0,n,m,k}^{(j)} = 0$ *for* $n \ge 1$ *and* $\mu_{r,n,m,k}^{(0)} = 1$ *for* $1 \le r \le n$.

Proof From (6) and (19), we have

$$\begin{aligned}
\mu^{(j)}_{r,n,m,k} &= \mu^{(j)}_{r-1,n,m,k} \\
&- \frac{jC_{r-1}}{\lambda\alpha(r-1)!\gamma_r}\int_{\beta}^{\infty} x^{j-1}[F(x)]^{\gamma_r-1}[\beta^{-\alpha}x^{\alpha-1}-x]f(x)g_m^{r-1}(F(x))dx \\
&= \frac{jC_{r-1}}{\lambda\alpha(r-1)!\gamma_r}\Big\{\int_{\beta}^{\infty} x^{j}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx \\
&- \beta^{-\alpha}\int_{\beta}^{\infty} x^{j+\alpha}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx\Big\}
\end{aligned}$$

and hence the result.

In particular, upon setting $r=1$ in Theorem 1, we deduce the following result.
Corollary 1. For the exponentiated Pareto type I distribution given in (4),

$$\left(1-\frac{j}{\lambda\alpha\gamma_1}\right)\mu^{(j)}_{1,n,m,k} = -\frac{j}{\alpha\lambda\beta^{\alpha}\gamma_r}\mu^{(j+\alpha)}_{1,n,m,k}. \tag{23}$$

Special Cases.
1) For $m=0$, $k=1$ in (22), we get the recurrence relations for ordinary order statistics,

$$\left(1-\frac{j}{\lambda\alpha(n-r+1)}\right)\mu^{(j)}_{n-r+1:n} = \mu^{(j)}_{n-r+2:n} - \frac{j}{\alpha\lambda\beta^{\alpha}(n-r+1)}\mu^{(j+\alpha)}_{n-r+1:n}.$$

Replacing $(n-r+1)$ by $(r-1)$, we have

$$\mu^{(j)}_{r:n} = \left(1-\frac{j}{\lambda\alpha(r-1)}\right)\mu^{(j)}_{r-1:n} + \frac{j}{\alpha\lambda\beta^{\alpha}(r-1)}\mu^{(j+\alpha)}_{r-1:n}.$$

2) For $m=-1$, $k\geq 1$ in (22) and (23), we get recurrence relations for k^{th} record values

$$\left(1-\frac{j}{\lambda\alpha k}\right)\mu^{(j)}_{L(r):k} = \mu^{(j)}_{L(r-1):k} - \frac{j}{\alpha\lambda\beta^{\alpha}k}\mu^{(j+\alpha)}_{L(r):k},$$

which is the result obtained by Kumar and Kumar (2017).
and

$$\left(1-\frac{j}{\lambda\alpha k}\right)\mu^{(j)}_{L(1):k} = -\frac{j}{\alpha\lambda\beta^{\alpha}\gamma_r}\mu^{(j+\alpha)}_{L(1)}.$$

Theorem 2. *For the distribution given in (4) and* $2\leq r\leq n$, $n\geq 2$ *and* $k=1,2,\ldots$

$$\mu^{(j)}_{r,n,m,k} = \mu^{(j)}_{r-1,n-1,m,k} - \frac{j(m+1)(r-1)}{\lambda\alpha\gamma_1\gamma_r}\Big\{\mu^{(j)}_{r,n,m,k} - \beta^{-\alpha}\mu^{(j+\alpha)}_{r,n,m,k}\Big\}. \tag{24}$$

Proof. Results can be established by considering (6) and (20).
Theorem 3. *For the distribution given in (4) and* $2\leq r\leq n$, $n\geq 2$ *and* $k=1,2,\ldots$

$$\mu^{(j)}_{r,n,m,k} = \mu^{(j)}_{r-1,n-1,m,k} - \frac{j}{\lambda\alpha\gamma_1}\Big\{\mu^{(j)}_{r,n,m,k} - \beta^{-\alpha}\mu^{(j)}_{r,n,m,k}\Big\}. \tag{25}$$

Proof. Results can be established by considering (6) and (21).

5 Relations for product moments

We shall first establish explicit expressions for the product moment of ith and jth generalized order statistics, $E\left[X'^{(i,j)}(r,s,n,m,k)\right] = \mu^{(i,j)}_{r,s,n,m,k}$. On using (10) and binomial expansion, the the explicit expressions for the product moments of DGOS $X'^{i}(r,n,m,k)$ and $X'^{j}(s,n,m,k)$, $1 \le r < s \le n$ can be obtained when $m \neq -1$ as

$$\begin{aligned}\mu^{(i,j)}_{r,s,n,m,k} &= \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1} (-1)^{u+v} \binom{r-1}{u}\binom{s-r-1}{v} \\ &\times \int_0^\infty x^i (F(x))^{(s-r+u-v)(m+1)-1} f(x) I(x) dx, \quad x > y \qquad (26)\end{aligned}$$

where

$$I(x) = \int_0^x y^j (F(y))^{\gamma_{s-v}-1} f(y) dy. \qquad (27)$$

by setting $t = [F(y)]^{1/\lambda}$ in (27), we obtain

$$I(x) = \beta^j \sum_{p=0}^{\infty} \frac{(j/\alpha)_{(p)} [F(x)]^{\gamma_{s-v}+p/\lambda}}{p!\,(\gamma_{s-v} + p/\lambda)}.$$

On substituting the above expression of $I(x)$ in (26), we find that

$$\begin{aligned}\mu^{(i,j)}_{r,s,n,m,k} &= \frac{\beta^j C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}\sum_{p=0}^{\infty} (-1)^{u+v} \binom{r-1}{u}\binom{s-r-1}{v} \\ &\times \frac{(j/\alpha)_p}{p!(\gamma_{s-v}+p/\lambda)} \int_\beta^\infty x^i [F(x)]^{\gamma_{s-v}+(s-r+u-v)(m+1)+p/\lambda-1} f(x) dx. \qquad (28)\end{aligned}$$

Again by setting $z = [F(x)]^{1/\theta}$ in (28) and simplifying the resulting equation, we get

$$\begin{aligned}\mu^{(i,j)}_{r,s,n,m,k} &= \frac{\beta^{i+j} C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} (-1)^{u+v} \\ &\times \binom{r-1}{u}\binom{s-r-1}{v} \frac{(j/\alpha)_p (i/\alpha)_q}{p!q!(\gamma_{s-v}+p/\lambda)(\gamma_{r-u}+(p+q)/\lambda)} \\ &= \frac{\beta^{i+j} C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(j/\alpha)_p (i/\alpha)_q}{p!q!} \\ &\times B\left(\frac{k}{m+1} + n - r + \frac{p+q}{\lambda(m+1)}, r\right) \\ &\times B\left(\frac{k}{m+1} + n - s + \frac{p}{\lambda(m+1)}, s - r\right) \qquad (29)\end{aligned}$$

Now using the relation,

$$\sum_{a=0}^{\infty} (-1)^a B(a+k, c) = B(k, c+b).$$

we have

$$\mu_{r,s,n,m,k}^{(i,j)} = \beta^{i+j} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(j/\alpha)_p (i/\alpha)_q}{p!q!} \frac{1}{\prod_{a=1}^{r} \left(1 + \frac{p+q}{\lambda\gamma_{r-a}}\right) \prod_{b=r+1}^{s} \left(1 + \frac{p}{\lambda\gamma_{s-b}}\right)}. \tag{30}$$

and when$m = -1$ that

$$\mu_{r,s,n,-1,k}^{(i,j)} = (\lambda k)^s \beta^{i+j} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(j/\alpha)_p (i/\alpha)_q}{p!q!(\lambda k + p + q)^r (\lambda k + p)^{s-r}}. \tag{31}$$

Special cases:
i) Putting $m = 0,\ k = 1$ in (30), the explicit formula for the product moments of lower order statistics of the $EPTI$ distribution can be obtained as

$$\begin{aligned} \mu_{n-r+1,n-s+1:n}^{(i,j)} &= \beta^{i+j} C_{r,s:n} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(j/\alpha)_p (i/\alpha)_q}{p!q!} \\ &\times \frac{1}{\prod_{a=1}^{r} \left(1 + \frac{p+q}{\lambda(n-r+a+1)}\right) \prod_{b=r+1}^{s} \left(1 + \frac{p}{\lambda(n-s+b+1)}\right)}. \end{aligned}$$

That is

$$\mu_{r,s:n}^{(i,j)} = \beta^{i+j} C_{r,s:n} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(j/\alpha)_p (i/\alpha)_q}{p!q!} \frac{1}{\prod_{a=1}^{r} \left(1 + \frac{p+q}{\lambda(s+a)}\right) \prod_{b=r+1}^{s} \left(1 + \frac{p}{\lambda(r+b)}\right)},$$

where

$$C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}.$$

ii)Putting $k = 1$ in (31), we deduce the explicit expression for the product moments of lower record values for the $EPTI$ distribution.

$$\mu_{r,s,n,-1,1}^{(i,j)} = \lambda^s \beta^{i+j} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(j/\alpha)_p (i/\alpha)_q}{p!q!(\lambda + p + q)^r (\lambda + p)^{s-r}},$$

as obtained by Kumar and Kumar (2017).

Making use of (6), we can derive recurrence relation for product moments of DGOS.

Theorem 4 *For the distribution given in (4) and for*$1 \le r < s \le n-1,\ n \ge 2$ *and* $k = 1, 2, ...$

$$\mu_{r,s,n,m,k}^{(i,j)} = \mu_{r,s-1,n,m,k}^{(i,j)} + \frac{j}{\lambda\alpha\gamma_s} \left\{\mu_{r,s,n,m,k}^{(i,j)} - \beta^{-\alpha} \mu_{r,s,n,m,k}^{(i,j+\alpha)}\right\}. \tag{32}$$

Proof. We have (Khan et al. (2008))

$$\begin{aligned} \mu_{r,s,n,m,k}^{(i,j)} &= \mu_{r,s-1,n,m,k}^{(i,j)} + \frac{jC_{s-1}}{\gamma_s (r-1)!(s-r-1)!} \int_\alpha^\beta \int_\alpha^x x^i y^{j-1} [F(x)]^m f(x) g_m^{r-1}(F(x)) \\ &\times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [F(y)]^{\gamma_s} dy dx, x > y. \end{aligned} \tag{33}$$

and hence (32), using (6) and (33).
Remark 1. Under the assumption given in Theorem 4 with $k = 1$, $m = 0$, we get the recurrence relation for product moment of DGOS Order statistics and at $k = 1$,$m = -1$, we deduce the recurrence relations for product moments of DGOS record values from $EPTI$ distribution.

6 Characterization

Theorem 5: *Let X be a non-negative random variable having an absolutely continuous distribution function $F(x)$ with $F(0) = 0$ and $0 < F(x) < 1$ for all $x > \beta$, then*

$$\left(1 - \frac{j}{\lambda\alpha\gamma_r}\right)\mu^{(j)}_{r,n,m,k} = \mu^{(j)}_{r-1,n,m,k} - \frac{j}{\alpha\lambda\beta^{\alpha}\gamma_r}\mu^{(j+\alpha)}_{r,n,m,k}. \tag{34}$$

if and only if

$$F(x) = [1 - (\beta/x)^{\alpha}]^{\lambda}, \quad x \geq \beta, \quad \alpha, \beta, \lambda > 0.$$

Proof: The necessary part follows immediately from equation (22). On the other hand if the recurrence relation in equation (34) is satisfied, then on using equation (6), we have

$$\begin{aligned}
&\frac{C_{r-1}}{(r-1)!}\int_{\beta}^{\infty} x^j[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx \\
&= \frac{(r-1)C_{r-1}}{\gamma_r(r-1)!}\int_{\beta}^{\infty} x^j[F(x)]^{\gamma_r+m}f(x)g_m^{r-2}(F(x))dx \\
&+ \frac{C_{r-1}}{\lambda\alpha\gamma_r(r-1)!}\int_{\beta}^{\infty} x^j[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx \\
&- \frac{j\beta^{-\alpha}C_{r-1}}{\lambda\alpha\gamma_r(r-1)!}\int_{\beta}^{\infty} x^{j+\alpha}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx
\end{aligned} \tag{35}$$

Now integrating the first integral on the right hand side of equation (35) by parts, we obtain

$$\begin{aligned}
&\frac{C_{r-1}}{(r-1)!}\int_{\beta}^{\infty} x^j[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx \\
&= \frac{jC_{r-1}}{\gamma_r(r-1)!}\int_{\beta}^{\infty} x^{j-1}[F(x)]^{\gamma_r}g_m^{r-1}(F(x))dx \\
&+ \frac{C_{r-1}}{(r-1)!}\int_{\beta}^{\infty} x^j[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx \\
&+ \frac{jC_{r-1}}{\lambda\alpha\gamma_r(r-1)!}\int_{\beta}^{\infty} x^j[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx \\
&- \frac{j\beta^{-\alpha}C_{r-1}}{\lambda\alpha\gamma_r(r-1)!}\int_{\beta}^{\infty} x^{j+\alpha}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx
\end{aligned}$$

which reduces to

$$\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_{\beta}^{\infty} x^{j-1}[F(x)]^{\gamma_r-1}g_m^{r-1}(F(x))\left\{F(x)-\frac{\beta^{-\alpha}x^{\alpha+1}}{\lambda\alpha}f(x)+\frac{x}{\lambda\alpha}\right\}dx=0. \quad (36)$$

Now applying a generalization of the Müntz-Szász Theorem (Hwang and Lin, 1984) to equation (36), we get

$$F(x)=\left[\frac{\beta^{-\alpha}x^{\alpha+1}}{\lambda\alpha}-\frac{x}{\lambda\alpha}\right]^{-1}$$

which proves that

$$F(x)=[1-(\beta/x)^{\alpha}]^{\lambda},\quad x\geq\beta,\quad \alpha,\beta,\lambda>0.$$

Theorem 6: Let X be a non-negative random variable having an absolutely continuous distribution function $F(x)$ with $F(0)=0$ and $0<F(x)<1$ for all $x>\beta$, then

$$\mu^{(j)}_{s,n,m,k|r,n,m,k}=\beta\sum_{p=0}^{\infty}\frac{(1/\alpha)_p[1-(\beta/x)^{\alpha}]^p}{p!\prod_{j=1}^{s-r}\left(1+\frac{p}{\lambda\gamma_{r+j}}\right)}. \quad (37)$$

if and only if

$$F(x)=[1-(\beta/x)^{\alpha}]^{\lambda},\quad x\geq\beta,\quad \alpha,\beta,\lambda>0.$$

Proof: To prove necessary part, from equation (21), we have

$$\begin{aligned}\mu^{(j)}_{s,n,m,k|r,n,m,k} &= \frac{C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^{s-r-1}}\\ &\times \int_{\beta}^{x} y\left[1-\left(\frac{F(y)}{F(x)}\right)^{m+1}\right]^{s-r-1}\left(\frac{F(y)}{F(x)}\right)^{\gamma_s-1}\frac{f(y)}{f(x)}dy. \end{aligned}\quad (38)$$

Setting $u=\frac{F(y)}{F(x)}$ from (5) in equation (38), we obtain

$$\begin{aligned}\mu^{(j)}_{s,n,m,k|r,n,m,k} &= \frac{\beta C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^{s-r-1}}\\ &\times \sum_{p=0}^{\infty}\frac{(1/\alpha)_p[1-(\beta/x)^{\alpha}]^p}{p!}\int_0^1(1-u^{m+1})^{s-r-1}u^{p/\lambda+\gamma_s-1}du. \end{aligned}\quad (39)$$

Put $t=u^{m+1}$ in (39) and simplifying the expression, we derive the relation in (37).
To prove sufficient part, we have from (37) and (38),

$$\frac{C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^{s-r-1}}\int_{\beta}^{x} y\left[(F(x))^{m+1}-(F(y))^{m+1}\right]^{s-r-1}(F(y))^{\gamma_s-1}f(y)dy$$

$$=[F(x)]^{\gamma_r+1}H_r(x), \quad (40)$$

where

$$H_r(x)=\beta\sum_{p=0}^{\infty}\frac{(1/\alpha)_p[1-(\beta/x)^{\alpha}]^p}{p!\prod_{j=1}^{s-r}\left(1+\frac{p}{\lambda\gamma_{r+j}}\right)}.$$

Differentiating both sides of (40) with respect to x, we get

$$\frac{C_{s-1}[F(x)]^m f(x)}{C_{r-1}(s-r-2)!(m+1)^{s-r-2}} \int_\beta^x y\left[(F(x))^{m+1}-(F(y))^{m+1}\right]^{s-r-2}(F(y))^{\gamma_s-1}f(y)dy$$

$$=[F(x)]^{\gamma_r+1}H_r'(x)+\gamma_{r+1}f(x)H_r(x).$$

In other words,

$$\gamma_{r+1}[F(x)]^{\gamma_{r+2}}f(x)H_{r+1}(x)=[F(x)]^{\gamma_{r+1}}H_r'(x)+\gamma_{r+1}[F(x)]^{\gamma_{r+1}-1}f(x)H_r(x).$$

Therefore

$$\frac{f(x)}{F(x)}=\frac{\lambda\alpha}{\beta^{-\alpha}x^{\alpha+1}-x},$$

whichn proves that

$$F(x)=[1-(\beta/x)^\alpha]^\lambda,\quad x\geq\beta,\quad \alpha,\beta,\lambda>0.$$

Remark 2. For $k=1$, $m=0$, we deduce the characterization results of lower order statistics and record values from $EPTI$ distribution, respectively.

7 Numerical Results

7.1 Tabulations of means and variances

The relations in (24) can be used to compute the expected values and second order moments of all order statistics and record values for sample sizes $n=1,2,3,\ldots,10$. In Tables 1 and 2, we have presented the expected values and variances of the rth order statistic from EPTI distribution for $n=1,2,\ldots,10$ and $\lambda=1.5,2$ and $\lambda=2.5,3.0$, respectively. One can see that the means and variances are decreasing with respect to n but increasing with respect to λ. In Table 3, we have shown the expected values and variances of upper record values from EPTI distribution for $n=1,2,\ldots,10$ and $\lambda=0.5,1,1.5$ and 2.0. We observe that the means and variances of upper record values are decreasing with respect to n but increasing with respect to λ.

Table 1: Expected values and variances of the rth order statistics from $EPTI$ distribution for $n = 1, 2, \ldots, 10$ and $\lambda = 1.5, 2$

λ	r	n	$E(X_{r:n})$	$Var(X_{r:n})$	λ	r	n	$E(X_{r:n})$	$Var(X_{r:n})$
1.5	1	1	1.3265	0.1321	2	1	1	1.3889	0.1543
		2	1.1649	0.0225			2	1.2114	0.0283
		3	1.1156	0.0096			3	1.1554	0.0129
		4	1.0910	0.0055			4	1.1267	0.0078
		5	1.0760	0.0037			5	1.1089	0.0054
		6	1.0659	0.0027			6	1.0965	0.0040
		7	1.0584	0.0020			7	1.0874	0.0032
		8	1.0527	0.0016			8	1.0803	0.0026
		9	1.0482	0.0013			9	1.0746	0.0022
		10	1.0446	0.0011			10	1.0699	0.0019
	2	2	1.4881	0.1894		2	2	1.5664	0.2173
		3	1.2635	0.0337			3	1.3234	0.0404
		4	1.1894	0.0147			4	1.2414	0.0183
		5	1.1510	0.0085			5	1.1980	0.0110
		6	1.1270	0.0056			6	1.1706	0.0075
		7	1.1104	0.0041			7	1.1514	0.0056
		8	1.0982	0.0031			8	1.1371	0.0044
		9	1.0888	0.0025			9	1.1259	0.0036
		10	1.0813	0.0021			10	1.1169	0.0030
	3	3	1.6004	0.2295		3	3	1.6880	0.2614
		4	1.3377	0.0417			4	1.4054	0.0491
		5	1.2471	0.0184			5	1.3064	0.0223
		6	1.1989	0.0107			6	1.2529	0.0133
		7	1.1684	0.0071			7	1.2186	0.0091
		8	1.1470	0.0052			8	1.1943	0.0068
		9	1.1312	0.0040			9	1.1761	0.0053
		10	1.1188	0.0032			10	1.1619	0.0043
	4	4	1.6880	0.2614		4	4	1.7822	0.2967
		5	1.3980	0.0481			5	1.4714	0.0561
		6	1.2953	0.0214			6	1.3598	0.0255
		7	1.2397	0.0125			7	1.2987	0.0153
		8	1.2039	0.0084			8	1.2590	0.0104
		9	1.1787	0.0061			9	1.2307	0.0077
		10	1.1599	0.0047			10	1.2094	0.0060
	5	5	1.7604	0.2885		5	5	1.8599	0.3266
		6	1.4494	0.0536			6	1.5271	0.0620
		7	1.3371	0.0240			7	1.4057	0.0283
		8	1.2754	0.0141			8	1.3383	0.0170
		9	1.2354	0.0095			9	1.2943	0.0116
		10	1.2070	0.0069			10	1.2627	0.0086
	6	6	1.8227	0.3122		6	6	1.9264	0.3530
		7	1.4943	0.0584			7	1.5757	0.0672
		8	1.3741	0.0262			8	1.4461	0.0308
		9	1.3074	0.0155			9	1.3736	0.0184
		10	1.2638	0.0104			10	1.3258	0.0126
	7	7	1.8774	0.3335		7	7	1.9849	0.3767
		8	1.5343	0.0627			8	1.6189	0.0719
		9	1.4075	0.0283			9	1.4824	0.0330
		10	1.3365	0.0168			10	1.4055	0.0198
	8	8	1.9264	0.3530		8	8	2.0371	0.3984
		9	1.5706	0.0666			9	1.6578	0.0762
		10	1.4379	0.0302			10	1.5154	0.0350
	9	9	1.9709	0.3710		9	9	2.0846	0.4184
		10	1.6037	0.0702			10	1.6935	0.0802
	10	10	2.0117	0.3878		10	10	2.1280	0.4371

Table 2: Expected values and variances of the rth order statistics from $EPTI$ distribution for $n = 1, 2, \ldots, 10$ and $\lambda = 2.5, 3$

λ	r	n	$E(X_{r:n})$	$Var(X_{r:n})$	λ	r	n	$E(X_{r:n})$	$Var(X_{r:n})$
2.5	1	1	1.4419	0.1731	3	1	1	1.4881	0.1894
		2	1.2520	0.0333			2	1.2882	0.0376
		3	1.1910	0.0157			3	1.2231	0.0181
		4	1.1592	0.0097			4	1.1889	0.0115
		5	1.1393	0.0069			5	1.1672	0.0083
		6	1.1253	0.0053			6	1.1519	0.0064
		7	1.1148	0.0043			7	1.1404	0.0053
		8	1.1067	0.0036			8	1.1314	0.0044
		9	1.1000	0.0031			9	1.1240	0.0038
		10	1.0945	0.0027			10	1.1179	0.0034
	2	2	1.6317	0.2408		2	2	1.6880	0.2614
		3	1.3742	0.0461			3	1.4185	0.0510
		4	1.2862	0.0214			4	1.3257	0.0240
		5	1.2392	0.0131			5	1.2757	0.0149
		6	1.2091	0.0091			6	1.2436	0.0105
		7	1.1880	0.0069			7	1.2209	0.0080
		8	1.1720	0.0055			8	1.2037	0.0064
		9	1.1596	0.0045			9	1.1902	0.0054
		10	1.1494	0.0038			10	1.1791	0.0046
	3	3	1.7604	0.2885		3	3	1.8227	0.3122
		4	1.4622	0.0553			4	1.5114	0.0606
		5	1.3567	0.0256			5	1.4006	0.0284
		6	1.2992	0.0155			6	1.3399	0.0174
		7	1.2621	0.0108			7	1.3005	0.0122
		8	1.2357	0.0081			8	1.2724	0.0093
		9	1.2157	0.0064			9	1.2510	0.0074
		10	1.2000	0.0052			10	1.2342	0.0061
	4	4	1.8599	0.3266		4	4	1.9264	0.3530
		5	1.5325	0.0627			5	1.5853	0.0685
		6	1.4142	0.0290			6	1.4613	0.0320
		7	1.3488	0.0176			7	1.3925	0.0196
		8	1.3061	0.0122			8	1.3474	0.0136
		9	1.2756	0.0091			9	1.3150	0.0103
		10	1.2524	0.0072			10	1.2904	0.0082
	5	5	1.9417	0.3591		5	5	2.0117	0.3878
		6	1.5917	0.0691			6	1.6473	0.0752
		7	1.4632	0.0319			7	1.5129	0.0350
		8	1.3915	0.0193			8	1.4376	0.0214
		9	1.3443	0.0134			9	1.3879	0.0149
		10	1.3103	0.0100			10	1.3520	0.0112
	6	6	2.0117	0.3878		6	6	2.0846	0.4184
		7	1.6431	0.0747			7	1.7010	0.0812
		8	1.5062	0.0345			8	1.5581	0.0378
		9	1.4292	0.0209			9	1.4773	0.0230
		10	1.3782	0.0145			10	1.4238	0.0160
	7	7	2.0731	0.4135		7	7	2.1485	0.4460
		8	1.6888	0.0797			8	1.7487	0.0865
		9	1.5448	0.0369			9	1.5984	0.0403
		10	1.4632	0.0223			10	1.5131	0.0245
	8	8	2.1280	0.4371		8	8	2.2056	0.4713
		9	1.7299	0.0843			9	1.7916	0.0915
		10	1.5797	0.0390			10	1.6350	0.0426
	9	9	2.1778	0.4589		9	9	2.2573	0.4947
		10	1.7674	0.0886			10	1.8308	0.0960
	10	10	2.2234	0.4792		10	10	2.3047	0.5165

Table 3: Expected values and variances of the upper record values from $EPTI$ distribution for $n = 1, 2, \ldots, 10$ and $\lambda = 1.5, 2.0, 2.5$ and 3.0

λ	n	$E(X)$	$Var(X)$	λ	n	$E(X)$	$Var(X)$
1.5	1	1.3265	0.1321	2	1	1.3889	0.1543
	2	1.1213	0.0162		2	1.1605	0.0211
	3	1.0593	0.0045		3	1.0865	0.0066
	4	1.0318	0.0016		4	1.0510	0.0026
	5	1.0178	0.0006		5	1.0315	0.0012
	6	1.0102	0.0002		6	1.0200	0.0005
	7	1.0060	0.0001		7	1.0128	0.0003
	8	1.0035	0.0000		8	1.0083	0.0001
	9	1.0021	0.0000		9	1.0055	0.0001
	10	1.0012	0.0000		10	1.0036	0.0000
2.5	1	1.4419	0.1731	3	1	1.4881	0.1894
	2	1.1955	0.0253		2	1.2273	0.0290
	3	1.1123	0.0086		3	1.1366	0.0104
	4	1.0705	0.0037		4	1.0896	0.0047
	5	1.0463	0.0018		5	1.0615	0.0024
	6	1.0312	0.0009		6	1.0433	0.0013
	7	1.0214	0.0005		7	1.0310	0.0007
	8	1.0149	0.0003		8	1.0225	0.0004
	9	1.0104	0.0001		9	1.0165	0.0002
	10	1.0073	0.0001		10	1.0121	0.0002

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