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GENERALISED RICCI SOLITONS ON TRANS SASAKIAN MANIFOLDS

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Abstract. The object of the present research is to show that a trans-Sasakian manifold which also satisfies the generalised gradient Ricci soliton equation, satisfying some conditions, is necessarily Einstein manifold.

Keywords and phrases: Generalised Ricci Solitons, Trans Sasakian manifold, Einstein manifold.

1. Introduction

In 1982, Hamilton [8] introduced that the Ricci solitons move under the Ricci flow simply by diffeomorphisms of the initial metric that is they are sationary points of the Ricci flow is given by

$$\frac{\partial g}{\partial t} = -2Ric(g). \tag{1.1}$$

Definition 1.1. *A Ricci soliton (g, V, λ) on a Riemannian manifold is defined by*

$$L_V g + 2S + 2\lambda = 0, \tag{1.2}$$

where S is the Ricci tensor, L_V is the Lie derivative along the vector field V on M and is a real scalar. Ricci soliton is said to be shrinking, steady or expanding according as $\lambda < 0, \lambda = 0$ and $\lambda > 0$, respectively.

If the vector field V is the gradient of a potential function -ψ , then g is called a gradient Ricci soliton and equation (1.2) assumes the form $Hess\psi = S + \lambda g$.

On the other hand, the roots of contact geometry lie in differential equations as in 1872 Sophus Lie introduced the notion of contact transformation as a geometric tool to study systems of differential equations. This subject has manifold connections with the other fields of pure mathematics, and substantial applications in applied areas such as mechanics, optics, phase space of dynamical system, thermodynamics and control theory.

The important of Ricci soliton comes from the fact that they are corresponding to self-similar solutions of the Ricci flow and at the same time they are natural generalizations of Einstein metrics. Some generalizations like gradient Ricci solitons [3], quasi Einstein manifolds [4] and generalized quasi Einstein manifolds [5] play an important roles in solutions of geometric flows and described the local structure of certain manifolds. In 2016, P. Nurowski and M. Randall [15] introduced the notion of genelalized Ricci soliton as a class of overdetermined system of equations

$$\mathcal{L}_X g + 2aX^\sharp \otimes X^\sharp - 2bS - 2\lambda g = 0 \tag{1.3}$$

where $\mathcal{L}_X g$ and $X^\sharp$ denotes the Lie derivative of the metric g in the directions of vector field X and the canonical 1-from associated to X, and some real constants a, b and λ .
In 1985, J. A. Oubina [16] introduced a new class of almost contact metric manifolds known as trans-Sasakian manifolds. This class consists of both Sasakian and Kenmotsu structures. The above manifolds are studied by several authors like D. E. Blair [1], J. C. Marrero [14], K. Kenmotsu [12]. In 1925, Levy [13] obtained the necessary and sufficient conditions for the existence of such tensors. later, R. Sharma [18] initiated the study of Ricci solitons in almost contact Riemannian geometry . Followed by M. M. Tripathi [20], G. Nagaraja et.al. [15] and M. Turan [19] and others extensively studied Ricci solitons in almost contact metric mnifolds. Therefore, motivated by these studies in the paresent paper author study the generalised Ricci soliton in trans-Sasakian manifolds. Trans-Sasakian manifolds arose in a natural way from the classification of almost contact metric structures and they appear as a natural generalization of both Sasakian and Kenmotsu manifolds

2. **Preliminaries**

Let a differentiable manifold M is said to be an almost contact metric manifold equipped with almost contact metric structure (ϕ, ξ, η, g) consisting of a $(1, 1)$ tensor field ϕ, a vector field ξ, a 1-form η and an indefinite metric g such that

$$\phi^2 X = -X + \eta(X)\xi, \ \ \eta(\xi) = 1, \ \ \eta \circ \phi = 0, \ \ \phi\xi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \ \ \eta(X) = g(X, \xi). \tag{2.2}$$

for all $X, Y \in M$, $\xi \in \Gamma(TM)$ and 1-form $\eta \in \Gamma(\bar{T}M)$.

In [21], Tanno classified the connected almost contact metric manifold. For such a manifold the sectional curvature of the plane section containing ξ is constant, say c. He showed that they can be divided into three classes. (1) homogeneous normal contact Riemannian manifolds with $c > 0$. Other two classes can be seen in Tanno [21].
In [9], Grey and Harvella was introduced the classification of almost Hermitian manifolds, there appears a class W_4 of Hermitian manifolds which are closely related to the conformal Kaehler manifolds. The class $C_6 \oplus C_5$ (*see* [1], [6], [14],) coincides with the class of trans-Sasakian structure of type (α, β). In fact, the local nature of the two sub classes, namely C_6 and C_5 of trans-Sasakian structures are characterized completely.
An almost conatct metric structure on M is called a trans-Sasakian (*see* [17], [14]) if $(M \times R, J, G)$ belongs to the class W_4, where J is the almost complex structure on $M \times R$ defined by

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi(X) - f\xi, \eta(X)\frac{d}{dt}\right)$$

for all vector fields X on M and smooth functions f on $M \times R$ and G is the product metric on $M \times R$. This may be expressed by the condition

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{2.3}$$

where α and β are some scalars functions. We note that trans Sasakian structure of type $(0,0)$, $(\alpha,0)$ and $(0,\beta)$ are the cosymplectic, α-Sasakian and β-Kenmotsu manifold respectively. In particular, if $\alpha=1,\beta=0$ and $\alpha=0,\beta=1$, then a trans-Sasakian manifold reduces to a Sasakian and Kenmotsu manifolds repectively. From (2.3), it follows that

$$\nabla_X\xi=-\alpha\phi(X)+\beta(X\eta(X)\xi, \tag{2.4}$$

and

$$(\nabla_X\eta)Y=-\alpha g(\phi X,Y)+\beta[g(X,Y)-\eta(X)\eta(Y)]. \tag{2.5}$$

for any vector fields X and Y on M, ∇ denotes the Levi-Civita connection with respect to g, α and β are smooth functions on M. The existence of condition (2.3) is ensure by the above discussion.

The Riemannian curvature tensor R with respect to Levi-Civita connections ∇ and the Ricci tenosr S of a Riemannian manifold M are defined by:

$$R(X,Y)Z=\nabla_X\nabla_YZ-\nabla_{YX}Z-\nabla_{[X,Y]}Z, \tag{2.6}$$

$$S(X,Y)=\sum_{i=1}^{n}g(R(X,e_i)e_i,Y) \tag{2.7}$$

for $X,Y,Z\in\Gamma(TM)$, where ∇ is with respect to the Riemannian metric g and $\{e_1,e_2,.....e_i\}$, where $1\leq i\leq n$ is the orthonormal frame.

Given a smooth function ψ on M, the gradient of ψ is defined by:

$$g(grad\psi,X)=X(\psi), \tag{2.8}$$

the *Hessian* of ψ is defined by:

$$(Hess\psi)(X,Y)=g(\nabla_X grad\psi,Y), \tag{2.9}$$

where $X,Y\in\Gamma(TM)$. For $X\in\Gamma(TM)$, we defined $X^\sharp\in\Gamma(\bar{T}M)$ by:

$$X^\sharp(Y)=g(X,Y). \tag{2.10}$$

The generalized Ricci soliton equation in Riemannian manifold M is defined by ([15]):

$$\mathcal{L}_Xg=-2aX^\sharp\odot X^\sharp+2bS+2\lambda g, \tag{2.11}$$

where $X\in\Gamma(TM)$ and $\mathcal{L}_Xg$ is the Lie-derivative of g along X given by:

$$(\mathcal{L}_Xg)(Y,Z)=g(\nabla_YX,Z)+g(\nabla_ZX,Y) \tag{2.12}$$

for all $Y,Z\in\Gamma(TM)$, and $a,b,\lambda\in R$. Equation (2.11), is a generalization of

(1) Killing's equation($a=b=\lambda=0$),
(2) Equation for homotheties ($a=b=0$),
(3) Ricci soliton ($a=0,b=-1$)'
(4) Case of Einstein-Weyl ($a=1,b=\frac{-1}{n-2}$),
(5) Metric projective structures with skew-symmetric Ricci tensor in projective class ($a=1,b=\frac{-1}{n-2},\lambda=0$),
(6) Vacuum near-horzion geometry equation ($a=1,b=\frac{1}{2}$), (*see* [2], [10], [11]).

Equation (2.11), is also a generalization of Einstein manifolds [5]. Note that, if $X = grad\psi$, where $\psi \in C^\infty(M)$, the generalised Ricci soliton equation is given by:

$$Hess\psi = -adf \odot df + bS + \lambda g. \tag{2.13}$$

3. Generalised Ricci solitons on $(M^n, \phi, \xi, \eta, g)$

In a n-dimensional trans Sasakian manifold M, we have the following relations:

$$R(X,Y)\xi = (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] + 2\alpha\beta[\eta(Y)\phi X - \eta(X)\phi Y] + [(Y\alpha)\phi X - (X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y] \tag{3.1}$$

$$S(X,\xi) = [((n-1)(\alpha^2 - \beta^2) - (\xi\beta)]\eta(X) + ((\phi X)\alpha) + (n-2)(X\beta), \tag{3.2}$$

$$Q\xi = (n-1)(\alpha^2 - \beta^2) - (\xi\beta))\xi + \phi(grad\alpha) - (n-2)(grad\beta), \tag{3.3}$$

where R is curvature tensor, while Q is the Ricci operator given by $S(X,Y) = g(QX,Y)$.
Further in an trans Sasakian manifold , we have

$$\phi(grad\alpha) = (n-2)(grad\beta), \tag{3.4}$$

and

$$2\alpha\beta + (\xi\alpha) = 0. \tag{3.5}$$

Using (3.4) and (3.5), for constants α and β , we have

$$R(\xi,X)Y = (\alpha^2 - \beta^2)[g(X,Y)\xi - \eta(Y)X] \tag{3.6}$$

$$R(X,Y)\xi = (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] \tag{3.7}$$

$$\eta(R(X,Y)Z) = (\alpha^2 - \beta^2)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] \tag{3.8}$$

$$S(X,\xi) = [((n-1)(\alpha^2 - \beta^2)]\eta(X) \tag{3.9}$$

$$Q\xi = [(n-1)(\alpha^2 - \beta^2)]\xi. \tag{3.10}$$

An important consequence of (2.4) is that ξ is a geodesic vector field.

$$\nabla_\xi \xi = 0. \tag{3.11}$$

For arbitrary vector field X, we have that

$$d\eta(\xi,X) = 0. \tag{3.12}$$

The ξ-sectional curvature K_ξ of M is the sectional curvature of the plane spanned by ξ and a unit vector field X. From (3.7) we have

$$K_\xi = g(R(\xi,X),\xi,X) = (\alpha^2 - \beta^2). \tag{3.13}$$

It follows from (3.13) that ξ-sectional curvature does not depend on X.

Theorem 3.1. *Let M be a trans-Sasakian manifold of dimenions n, and satisfies the generalised Ricci soliton (2.13) with condition $a[\lambda+(n-1)b(\alpha^2-\beta^2)] \neq -1$, then ψ is a constant function. Furthermore, if $b \neq 0$, then M is an Einstein manifold.*

From Theorem 3.1, we get the following remarks:

Remark 3.2. *let M is a trans-Sasakian maifold and satisfies the gardient Ricci soliton equation $Hess\psi = -S + \lambda g$, then ψ is a constant function and M is an Einstein manifold.*

Remark 3.3. *In a trans-Sasakian manifold M, there is no non-constant smooth function ψ, such that $Hess\psi = \lambda g$, for some constant λ.*

For the proof of the Theorem 3.1, first we need to prove the following lemmas.

Lemma 3.4. *Let M be a trans-Sasakian manifold. Then we have*

$$(\mathcal{L}_\xi(\mathcal{L}_X g))(Y,\xi) = (\alpha^2-\beta^2)g(X,Y) + g(\nabla_\xi\nabla_\xi X, Y) + Yg(\nabla_\xi X,\xi), \tag{3.14}$$

where $X, Y \in \Gamma(TM)$ and Y is orthogonal to ξ.

Proof. From the property of Lie-derivative we note that

$$(\mathcal{L}_\xi(\mathcal{L}_X g))(Y,\xi) = \xi((\mathcal{L}_X g)(Y,\xi)) - (\mathcal{L}_X g)(\mathcal{L}_\xi Y,\xi) - (\mathcal{L}_X g)(Y,\mathcal{L}_\xi\xi), \tag{3.15}$$

since $\mathcal{L}_\xi Y = [\xi, Y]$, $\mathcal{L}_\xi\xi = [\xi,\xi]$, by using (2.12) and (3.15), we have

$$\begin{aligned}(\mathcal{L}_\xi(\mathcal{L}_X g))(Y,\xi) &= \xi g(\nabla_Y X,\xi) + \xi g(\nabla_\xi X, Y) - g(\nabla_{[\xi,Y]}X,\xi) \\ &\quad - g(\nabla_\xi X,[\xi,Y]) \\ &= g(\nabla_\xi\nabla_Y X,\xi) + g(\nabla_Y X,\nabla_\xi\xi) + g(\nabla_\xi\nabla_\xi X, Y) \\ &\quad + g(\nabla_\xi X,\nabla_\xi Y) - g(\nabla_\xi X,\nabla_\xi Y) - g(\nabla_{[\xi,Y]}X,\xi) + g(\nabla_\xi X,\nabla_Y\xi),\end{aligned} \tag{3.16}$$

from (2.4), we get $\nabla_\xi\xi = \phi\xi = 0$, so that we get

$$\begin{aligned}(\mathcal{L}_\xi(\mathcal{L}_X g))(Y,\xi) &= g(\nabla_\xi\nabla_Y X,\xi) + g(\nabla_\xi\nabla_\xi X, Y) - g(\nabla_{[\xi,Y]}X,\xi) \\ &\quad + Yg(\nabla_\xi X,\xi) - g(\nabla_Y\nabla_\xi X,\xi),\end{aligned} \tag{3.17}$$

using (3.6) and (3.17), we obtain

$$(\mathcal{L}_\xi(\mathcal{L}_X g))(Y,\xi) = g(R(\xi,Y)X,\xi) + g(\nabla_\xi\nabla_\xi X, Y) + Yg(\nabla_\xi X,\xi). \tag{3.18}$$

Now from (3.6), with $g(Y,\xi) = 0$, we get

$$g(R(\xi,Y)X,\xi) = g(R(Y,\xi)\xi, X) = (\alpha^2-\beta^2)g(X,Y). \tag{3.19}$$

the Lemma follows from (3.17) and (3.18).

□

Now, we have another useful lemma

Lemma 3.5. *Let M be a Riemannian manifold, and let $\psi \in C^\infty(M)$. Then we have*

$$(\mathcal{L}_\xi(df\odot df))(Y,\xi) = Y(\xi(\psi))\xi(\psi) + Y(\psi)\xi(\xi(\psi)),$$

where $\xi, Y \in \Gamma(TM)$.

Proof. We calculate:

$$\begin{aligned}(\mathcal{L}_\xi(df\odot df))(Y,\xi) &= \xi(Y(\psi)\xi(\psi) - [\xi,Y](\psi)\xi(\psi) - Y(\psi)[\xi,\xi](\psi) \\ &= \xi(Y(\psi))\xi(\psi) + Y(\psi)\xi(\xi(\psi)) - [\xi,Y](\psi)\xi(\psi),\end{aligned}$$

since $[\xi,Y](\psi) = \xi(Y(\psi)) - Y(\xi(\psi))$, we get

$$\begin{aligned}(\mathcal{L}_\xi(df\odot df))(Y,\xi) &= [\xi,Y](\psi)\xi(\psi) + Y(\xi(\psi))\xi(\psi) + Y(\psi)\xi(\xi(\psi)) - [\xi,Y](\psi)\xi(\psi) \\ &= Y(\xi(\psi))\xi(\psi) + Y(\psi)\xi(\xi(\psi)).\end{aligned}$$

□

Lemma 3.6. *Let M be a trans-Sasakian manifold of dimension n, and satisfies the generalised Ricci soliton equation (2.13). Then we have*

$$\nabla_\xi\ grad\psi = [\lambda + b(n-1)(\alpha^2-\beta^2)]\xi - a\xi(\psi)grad\psi. \tag{3.21}$$

Proof. Let $Y \in \Gamma(TM)$, form the definition of Ricci curvature S (2.7) , and the curvature condition (3.7), we have

$$\begin{aligned} S(X,Y) &= g(R(\xi,e_i)e_i,Y) \\ &= g(R(e_i,Y)\xi,e_i) \\ &= (\alpha^2-\beta^2)[\eta(Y)g(e_i,e_i)-\eta(e_i)g(X,e_i) \\ &= (\alpha^2-\beta^2)[n\eta(Y)-\eta(Y)] \\ &= (n-1)(\alpha^2-\beta^2)\eta(Y) = (n-1) \\ &= (n-1)(\alpha^2-\beta^2)g(\xi,Y), \end{aligned}$$

where $\{e_1, e_2, ...e_i\}$, and $1 \leq i \leq n$ is an orthonormal frame on M, implies that:

$$\begin{aligned} \lambda g(\xi,Y)+bS(\xi,Y) &= \lambda g(\xi,Y)+b(n-1)(\alpha^2-\beta^2)g(\xi,Y) \\ &= [\lambda+b(n-1)(\alpha^2-\beta^2)]g(\xi,Y), \end{aligned} \tag{3.22}$$

from (2.13) and (3.22), we obtain

$$\begin{aligned} (Hess\psi)(\xi,Y) &= -a\xi(\psi)(Y)(\psi)+[\lambda+b(n-1)(\alpha^2-\beta^2)]g(\xi,Y) \\ &= -a\xi(\psi)g(grad\psi,Y)+[\lambda+b(n-1)(\alpha^2-\beta^2)]g(\xi,Y), \end{aligned} \tag{3.23}$$

the Lemma follows from equation (3.23) and the definition of *Hessian* (2.8). □

Now, with help of Lemma 3.4 , Lemma 3.5 and Lemma 3.6 we can prove the Theorem 3.1 .

Proof. Theorem 3.1. Let $Y \in \Gamma(TM)$, such that $g(\xi, Y) = 0$, from Lemma 3.4, with $X = grad\ \psi$, we have

$$2(\mathcal{L}_\xi(Hess\psi))(Y,\xi) = Y(\psi)+g(\nabla_\xi\nabla_\xi\ grad\psi,Y)+Yg(\nabla_\xi\ grad\psi,\xi), \tag{3.24}$$

from Lemma 3.6, and equation (3.24), we get

$$\begin{aligned} 2(\mathcal{L}_\xi(Hess\psi))(Y,\xi) = Y(\psi)&+[\lambda+b(n-1)(\alpha^2-\beta^2)]g(\nabla_\xi,Y)-ag(\nabla_\xi(\xi(\psi)grad\,\psi),Y) \\ &+[\lambda+b(n-1)(\alpha^2-\beta^2)]Yg(\xi,\xi)-aY(\xi(\psi)^2), \end{aligned} \tag{3.25}$$

since $\nabla_\xi\xi = 0$ and $g(\xi, \xi) = 1$, from equation (3.25), we obtain:

$$\begin{aligned} 2(\mathcal{L}_\xi(Hess\psi))(Y,\xi) = Y(\psi)&-a\xi(\xi(\psi))Y(\psi)-a\xi(\psi)g(\nabla_\xi\ grad\psi,Y) \\ &-2a\xi(\psi)Y(\xi(\psi)). \end{aligned} \tag{3.26}$$

From Lemma 3.6, equation (3.26), and since $g(\xi, Y) = 0$, we have

$$\begin{aligned} 2(\mathcal{L}_\xi(Hess\psi))(Y,\xi) = Y(\psi)&-a\xi(\xi(\psi))Y(\psi)+a^2\xi(\psi)^2Y(\psi) \\ &-2a\xi(\psi)Y(\xi(\psi)). \end{aligned} \tag{3.27}$$

Note that, from (2.4) and (2.5), we have $\mathcal{L}_\xi g = 0$, which is a Killing vector filed, implies that $\mathcal{L}_\xi S = 0$, taking the Lie derivative to the generalised Ricci soliton equation (2.13) yields

$$\begin{aligned} Y(\psi)&-a\xi(\xi(\psi))Y(\psi)+a^2\xi(\psi)^2Y(\psi)-2a\xi(\psi)Y(\xi(\psi)) \\ &= -2aY(\xi(\psi))\xi(\psi)-2aY(\psi)\xi(\xi(\psi)), \end{aligned} \tag{3.28}$$

is equivalent to

$$Y(\psi)[1+a\xi(\xi(\psi))+a^2\xi(\psi)^2] = 0, \tag{3.29}$$

according to Lemma 3.6, we have

$$a\xi(\xi(\psi)) = a\xi g(\xi, grad\ \psi) \tag{3.30}$$
$$= ag(\xi, \nabla_\xi\ grad\psi)$$
$$= a[\lambda + b(n-1)(\alpha^2 - \beta^2)] - a^2\xi(\psi)^2,$$

by equations (3.29) and (3.30), we obtain

$$Y(\psi)[\lambda + b(n-1)(\alpha^2 - \beta^2)] = 0, \tag{3.31}$$

since $[\lambda + b(n-1)(\alpha^2 - \beta^2)] \neq -1$, we find that $Y(\psi) = 0$, i.e., $grad\psi$ is parallel to ξ. Hence $grad\ \psi = 0$ as $D = ker\eta$ is not integrable any where, which means ψ is a constant function. □

Now, for particular values of α and β we have follwoing cases:
For $\alpha = 0$ or $(\beta = 1)$, we can state:

Corollary 3.7. *Let M be a β-Kenmotsu (or Kenmotsu) manifold of dimenions n, and satisfies the generalised Ricci soliton (2.13) with condition $a[\lambda - (n-1)b\beta^2)] \neq -1$, then ψ is a constant function. Furthermore, if $b \neq 0$, then M is an Einstein manifold.*

For $\beta = 0$, or $(\alpha = 1)$ we can state:

Corollary 3.8. *Let M be a α-Sasakian (or Sasakian) manifold of dimenions n, and satisfies the generalised Ricci soliton (2.13) with condition $a[\lambda + (n-1)b\alpha^2)] \neq -1$, then ψ is a constant function. Furthermore, if $b \neq 0$, then M is an Einstein manifold.*

References

[1] Blair, D. E. and Oubina, J. A., Conformal and related changes of metric on the product of two almost contact metric manifolds, Publ. Mat. 34 (1990), 199-207.
[2] Chrusciel,P. T., Reall, H. S. and Tod, P., On non-existence of static vacuum black holes with degenerate components of the event horizon, Classical Quantum Gravity, 23(2006), 549554.
[3] Cao, H. D. Recent progress on Ricci solitons, Adv. Lect. Math. (ALM), 11, 138, (2009).
[4] Case, J. S. Shu, Y., Wei, G., Rigidity of quasi Einstein metrics, Diff. Geo. Appl., 20, 93100, (2011).
[5] Catino, G., Generalized quasi Einstein manifolds with harmonic Weyl tensor, Math. Z., 271, 751756, (2012).
[6] Chinea, D., Gonzales, C., A classification of almost contact metric manifolds, Ann. Mat. Pura Appl. 156 (1990) 15-30.
[7] Friedan, D., Non linear models in $2 + \epsilon$ dimensions, Ann. Phys. 163 (1985) 318-419.
[8] Hamilton, R. S., The Ricci flow on surfaces, Mathematics and general relativity, (Santa Cruz. CA, 1986), Contemp. Math. 71, Amer. Math. Soc., (1988), 237-262.
[9] Gray, A. and Harvella, L. M., The sixteen classes of almost Hermitian manifolds and their linear invariants, Ann. Mat. Pura Appl., 123(4) (1980), 35-58.
[10] Jezierski, J., On the existence of Kundt's metrics and degenerate (or extremal) Killing horizons, Classical Quantum Gravity, 26(2009), 035011, 11 pp.
[11] Kunduri, H. K and J. Lucietti, J., Classification of near-horizon geometries of extremal black holes, Living Rev. Relativity, 16(8)(2013).
[12] Kenmotsu, K., A class of almost contact Riemannian manifolds, Tohoku Math. J. 24(2) (1972), 93-103.
[13] Levy, H. Symmetric tensors of the second order whose covariant derivatives vanish, Ann. Math. 27(2) (1925), 91-98.
[14] Marrero, J. C., The local structure of Trans-Sasakian manifolds, Annali di Mat. Pura ed Appl. 162 (1992), 77-86.

[15] Nurowski, P. and Randall, M., Generalized Ricci solitons, J. Geom. Anal., 26, 1280 1345 (2016).

[16] Nagaraja, H.G. and C.R. Premalatha, C. R., Ricci solitons in Kenmotsu manifolds, J. Math. Anal. 3 (2) (2012), 18-24.

[17] Oubina, J. A., New classes of almost contact metric structures, Publ. Math. Debrecen 32 (1985), 187-193.

[18] Sharma, R., Certain results on K-contact and (k,μ)-contact manifolds, J. Geom., 89(1-2) (2008), 138-147.

[19] Turan, M., De, U. C. and Yildiz, A., Ricci solitons and gradient Ricci solitons on 3- dimensional trans-Sasakian manifolds, Filomat, 26(2) (2012), 363-370.

[20] Tripathi, M. M., Ricci solitons in contact metric manifolds, arXiv:0801.4222 [math.DG].

[21] Tanno, S., The automorphism groups of almost contact Riemannian manifolds,Tohoku Math.J. 21 (1969),21-38.